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Financial Markets and the Real Economy: A Statistical Field Perspective on Capital Allocation and Accumulation

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Pierre Gosselin, Aïleen Lotz, Marc Wambst

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Pierre Gosselin[*]   Aïleen Lotz[†]   Marc Wambst[‡]

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Abstract

This paper provides a general method to directly translate a classical economic framework with a large number of agents into a field-formalism model. This type of formalism allows the analytical treatment of economic models with an arbitrary number of agents, while preserving the system's interactions and microeconomic features of the individual level.

We apply this methodology to model the interactions between financial markets and the real economy, described in a classical framework of a large number of heterogeneous agents, investors and firms. Firms are spread among sectors but may shift between sectors to improve their returns. They compete by producing differentiated goods and reward their investors by paying dividends and through their stocks' valuation. Investors invest in firms and move along sectors based on firms' expected long-run returns.

The field-formalism model derived from this framework allows for collective states to emerge. We show that the number of firms in each sector depends on the aggregate financial capital invested in the sector and its firms' expected long-term returns. Capital accumulation in each sector depends both on short-term returns and expected long-term returns relative to neighbouring sectors.

For each sector, three patterns of accumulation emerge. In the first pattern, the dividend component of short-term returns is determinant for sectors with small number of firms and low capital. In the second pattern, both short and long-term returns in the sector drive intermediate-to-high capital. In the third pattern, higher expectations of long-term returns drive massive inputs of capital.

Instability in capital accumulation may arise among and within sectors. We therefore widen our approach and study the dynamics of the collective configurations, in particular interactions between average capital and expected long-term returns, and show that overall stability crucially depends on the expectations' formation process.

Expectations that are highly reactive to capital variations stabilize high capital configurations, and drive low-to-moderate capital sectors towards zero or a higher level of capital, depending on their initial capital. Inversely, low-to moderate capital configurations are stabilized by expectations moderately reactive to capital variations, and drive high capital sectors towards more moderate level of capital equilibria.

Eventually, the combination of expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital imply that large fluctuations of capital in the system, at the possible expense of the real economy.

Key words: Financial Markets, Real Economy, Statistical Field Theory, Phase Transition, Capital Allocation, Exchange Space, Multi-Agent Model, Interaction Agents.

JEL Classification: B40, C02, C60, E00, E1, G10

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[*]Pierre Gosselin : Institut Fourier, UMR 5582 CNRS-UGA, Université Grenoble Alpes, BP 74, 38402 St Martin d'Hères, France. E-Mail: Pierre.Gosselin@univ-grenoble-alpes.fr

[†]Aïleen Lotz: Cerca Trova, BP 114, 38001 Grenoble Cedex 1, France. E-mail: a.lotz@cercatrova.eu

[‡]Marc Wambst : IRMA, UMR 7501 CNRS, Université de Strasbourg, 7 rue René Descartes, 67084, Strasbourg Cedex, France. E-Mail: wambst@math.unistra.fr

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1   Introduction

This paper studies the interactions between financial and physical capital. A large number of heterogeneous agents is divided into two groups, investors and firms. The specificity of our work is to model these interactions between agents as a field-theoretic model.

The field-formalism used in this paper is rooted in a probabilistic description of economic systems with large number of agents. Classically, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations. The same system of agents can however be seen as a probabilistic: indeed an agent can be described by a *probability density* that is, due to idiosyncratic uncertainties, centred around the classical optimal path[1] (see Gosselin, Lotz and Wambst 2017, 2020, 2021). In this probabilistic approach, each possible dynamics for the set of N agents must be taken into account and weighted by its probability. The system is then described by a *statistical weight*, the probability density for any configuration of N arbitrary individual paths. Once this statistical weight found, we can compute the transition probabilities of the system, i.e. the probabilities for any number of agents to evolve from an initial to a final state in A_i, B_i in a given time.

Because this probabilistic approach implies to keep track of the N agents' probability transitions, it is practically untractable for a large number of agents. It remains however a necessary step, since it can conveniently be translated into a more compact *field formalism* (see Gosselin, Lotz and Wambst 2017, 2020, 2021). This field formalism preserves the essential information encoded in the model but implements a change in perspective. It does not keep track of the N-indexed agents, but describes their dynamics and interactions as a collective thread of all possible anonymous paths. This collective thread can be seen as an environment that itself conditions the dynamics of individual agents from one state to another. The field formalism eases the computation of transition functions. More importantly, it detects the collective states or phases encompassed in the field, that would otherwise remain indetectable using the probabilistic formulation.

To translate the probabilistic approach into a field model, the N agents' trajectories $\mathbf{A}_i(t)$ are replaced by a field Ψ, which is a complex valued function that solely depends on a single set of variables, $\mathbf{A}$. The statistical weight of the probabilistic approach is translated into a probability density on the space of complex-valued functions of the variables $\mathbf{A}$. For the configuration $\Psi(\mathbf{A})$, this probability density has the form $\exp(-S(\Psi))$. The functional $S(\Psi)$ is called the *field action functional*. It encodes the microscopic features of individual agents dynamics and interactions. The idea is that of a dictionnary that would translate the various terms of the classical description in terms of their field equivalent. The integral of $\exp(-S(\Psi))$ over the configurations Ψ is the *partition function* of the system. The fields that minimize the action functional are the *classical background fields,* or more simply the *background fields*. They encapsulate the collective states of the system.

For several types of agents, the generalisation is straightforward. Each type α is described by a field $\Psi_\alpha(\mathbf{A}_\alpha)$. The field action depends on the whole set of fields $\{\Psi_\alpha\}$. It accounts for all types of agents and their interactions, and writes $S(\{\Psi_\alpha\})$. The form of $S(\{\Psi_\alpha\})$ is obtained directly from the classical description of our model.

We have detailed in previous papers the transitions from the classical to the probabilistic frameworks, and then from probabilistic to field models, and studied individual behaviors in given collective states.

In this paper, we present a shortcut to directly translate a classical framework into a field-formalism model and find the backgound fields that describe the collective states and the global features of the system.

Two groups of agents, producers and investors, each represent the real economy and the financial markets, respectively. The first group is composed of a large number of firms in different sectors that collectively own the entire physical capital. The second group, investors, holds the entire financial capital and allocate it between firms across sectors according to investment preferences, expected returns, and stock prices variations on financial markets. In return, firms pay their investors dividends. Thus financial capital is a function of dividends and stocks' valuations, whereas physical capital is a function of the overall capital allocated by the financial sector. They are described by two different interacting fields and by the system action functional. The solutions to the minimization equations of the action functional are the background fields of the system. They characterize the collective states of the system, structure the interactions between the two types of agents, and condition individual dynamics. From a sector perspective, the collective states determine the

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[1] Due to the infinite number of possible paths, each individual path has a null probability to exist. We therefore use the word probability density rather than probability.

capital average distribution and the firms' concentration within sectors, given external parameters, such as changes in expected returns, technological advances, as well as their dynamics when these external conditions evolve.

We first show that the number of firms per sector depends on the average level of capital invested in this sector and on its expected long-term return, i.e. the long-term return relative to those of neighbouring sectors. Sectoral capital accumulation itself depends on short-term returns, both absolute and relative, and on relative expected long-term returns.

Equilibrium capital at each point of the sector space characteristizes the collective configuration of the system. This equilibrium may however be unstable: due to the limited number of agents, changes in parameters or expectations may induce changes in portfolio allocation that may leave some sectors deserted. Both the sectoral equilibrium capital and number of agents must therefore be interpreted as potential, not actual, equilibria: they are thresholds. At a macro-timescale, any deviation from an equilibrium drives the sector towards the next stable equilibrium, zero included, and if there is none, towards infinity. Note that this looming potential instability in sectors depends on the position of the sector relative to its neighbouring sectors. This notion of instability is thus relative and context-dependent: variations of parameters in some sectors may propagate to other sectors.

To account for this systemic instability, we adopt a wider approach to our model: we consider a dynamic system involving average capital and endogenized long-term expected returns, that is the most volatile parameter of our model. In such a dynamic system, average capital per sector interact with one another, but also with long-term expected returns.

This dynamic system differs from those in standard economic: whereas in economics the dynamics is usually studied around a static equilibria, we consider the dynamic interactions between potential equilibria and expected long-term returns.

Some solutions of this dynamic system are oscillatory: changes in one or several sectors may propagate over the whole sectors' space. We find, for each sector, the conditions of stable or unstable oscillations for the system. Depending on the sector's specific characteristics, oscillations in average capital and expected long-term returns may dampen or increase. Some characteristic of the system discriminate stable and unstable oscillations: some formation of expectations favour overall stability in equilibria, others deter it.

Eventually, fluctuations in financial expectations impose their pace to the real economy. The combination of expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital imply that large fluctuations of capital in the system, at the possible expense of the real economy.

The paper is organized as follows. The second section is a literature review. Section three presents the microeconomic framework on which our field model is based. Section four presents the general method of translation of a model with a large number of agents into a field theoretic model. This method is applied to our microeconomic framework in section five to derive the field-theoretic representation of the system. Section 6 exposes the use of the field model in our context, and the various averages it allows to compute. In section 7, we present the resolution of the model. We derive the background field for the real economy and the density of firms par sector. We then compute the background field for the financial agents and find the density of investors per sector and the defining equation for average capital per firm per sector. Section 8 investigates the solutions of this equation. It studies its differential form, expands it around some particular solutions, and finds directly the solutions for some particular forms of the parameter functions defining the system. In section 9, the model is extended to a dynamic system at the macro-time scale by endogenizing the expected long-term revenue. This dynamic system presents some oscillatory solutions whose stability depend on the various patterns of accumulation. Section 10 gathers the interpretations of the model. The results are presented in section 11 and discussed in Section 12. Section 13 concludes.

2   Literature review

Several branches of the economic literature seek to replace the representative agent by a collection of heterogeneous ones. Among other things, they differ in the way they model this collection of agents.

A first branch of the literature represent this collection of agents by probability densities. This is the approach followed by mean field theory, heterogeneous agents new Keynesian (HANK) models and the information theoretic approach to economics.

Mean field theory studies the evolution of agents' density in the state space of economic variables. It includes the interactions between agents and the population as a whole, but does not consider the direct interactions between agents. This approach is thus at an intermediate scale between the macro and micro scale: it does not aggregate agents but replace them by an overall probability distribution. Mean field theory has been applied to game theory (Bensoussan et al. 2018, Lasry et al. 2010a, b) and economics (Gomes et al. 2015). However, these mean fields are actually probability distributions. In our formalism, the notion of fields refers to some abstract complex functions defined on the state space and similar to the second-quantized-wave functions of quantum theory. Besides, our formalism directly includes the interactions between agents at the individual level.

Heterogeneous agents new Keynesian (HANK) models uses a probabilistic treatment similar to mean fields theory. An equilibrium probability distribution is derived from a set of optimizing heterogeneous agents in a new Keynesian context (see Kaplan and Violante 2018 for an account). Our approach, on the contrary, focuses on the direct interactions between agents at the microeconomic level. We do not look for an equilibrium probability distribution for each agent, but rather directly build a probability density for the system of N agents seen as a whole, that includes interactions, and then translate this probability density in terms of fields. The states' space we consider is thus much larger than those considered in the above approaches. Because it is the space of all paths for a large number of agents, it allows to study the agents' economic structural relations and the emergence of the particular phases or collective states induced by these specific micro-relations, that will in turn impact each agent's stochastic dynamics at the microeconomic level. Other differences are worth mentioning. While HANK models stress the role of an infinite number of heterogeneously-behaved consumers, our formalism dwells on the relations between physical and financial capital[2]. Besides, our formalism does not rely on agents' rationality assumptions, since for a large number of agents, behaviours, be they fully or partly rational, can be modelled as random.

The information theoretic approach to economics (see Yang 2018) considers probabilistic states around the equilibrium. It is close to our methodological stance: it replaces the Walrasian equilibrium by a statistical equilibrium derived from a entropy maximisation program. Our statistical weight is similar to the one they use, but is directly built from microeconomic dynamic equations. The same difference stands for the rational inattention theory (Sims 2006) in which non-gaussian density laws are derived from limited information and constraints: our setting directly includes constraints in the random description of an agent (Gosselin, Lotz, Wambst 2020).

A second branch of the literature is closest to our approach, since it considers the interacting system of agents in itself. It is the multi-agent systems literature, notably agent based models (see Gaffard Napoletano 2012, Mandel et al. 2010 2012) and economic networks (Jackson 2010).

Agent-based models use general macroeconomics models, whereas network models lower-scale models such as contract theory, behaviour diffusion, information sharing or learning. In both settings, agents are typically defined by and follow various sets of rules, leading to the emergence of equilibria and dynamics otherwise inaccessible to the representative agent set-up. Both approaches are however highly numerical and model-dependent and relies on microeconomic relations - such as ad-hoc reaction functions - that may be too simplistic. Statistical fields theory on the contrary accounts for transitions between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents: they are grounded in behaviours and interactions structures. Describing these structures in terms of field theory allows for the emergence of phases at the macro scale, and the study of their impact at the individual level.

A third branch of the literature, Econophysics, is also related to ours, since it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a,b and references therein; or Lux 2008, 2016). But it tends to focus on empirical laws, rather than apply the full potential of field theory to economic systems. In the same vein, Kleinert (2009) uses path integrals to model stock prices' dynamics. Our approach in contrast keeps track of usual microeconomic concepts, such as utility functions, expectations and forward-looking behaviours and includes these behaviours into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems.

The literature on interactions between finance and real economy or capital accumulation takes place mainly in the context of DGSE models. (for a review of the literature, see Cochrane 2006; for further developments see Grassetti et al. 2022, Grosshans and Zeisberger 2018, Böhm et al. 2008, Caggese and

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[2]Note that our formalism could also include heterogeneous consumers (see Gosselin, Lotz, Wambst 2020).

Orive, Bernanke e al. 1999, Campello et al. 2010, Holmstrom and Tirole 1997, Jermann, and Quadrini 2012, Khan Thomas 2013, Monacelli et al. 2011). Theoretical models include several types of agents at the aggregated level. They describe the interactions between a few representative agents such as producers for possibly several sectors, consumers, financial intermediaries, etc. to determine interest rates, levels of production, asset pricing, in a context of ad-hoc anticipations.

Our formalism differs from this literature in three ways. First, we consider several groups of large number of agents to describe the emergence of collective states and study the continuous space of sectors. Second, we consider expected returns and the longer-term horizon as somewhat exogeneous or structural. Expected returns are a combination of elements, such as technology, returns, productivity, sectoral capital stock, expectations and beliefs. These returns are also a function defined over the sectors' space: the system's background fields are functionals of these expected returns. Taken together, the background fields of a field model describe an economic environment for a given configuration of expected returns. As such, expected returns are at first seen as exogenous. It is only in a second step, when we consider the dynamics between capital accumulation and expectations, that expectations may themselves be seen as endogenous. Even then, the form of relations between actual and expected variables specified are general enough to derive some types of possible dynamics.

Last but not least, we do not seek individual or even aggregated dynamics, but rather background fields that describe potential long-term equilibria and may evolve with the structural parameters. For such a background, agents' individual typical dynamics may nevertheless be retrieved through Green functions (see GLW). These functions compute the transition probabilities from one capital-sector point to another. But backgrounds themselves may be considered as dynamical quantities. Structural or long-term variations in the returns' landscape may modify the background, and in turn the individual dynamics. Expected returns themselves depend on, and interact with, capital accumulation.

3 The Microeconomic Framework

This section develops a microeconomic framework that will be turned into a field model. Since our goal is to picture the interactions between the real and the financial economy, we consider two groups of agents, producers and investors. In the following, we will refer to producers or firms i indistinctively, and use the upper script ˆ for variables describing investors.

3.1 Producers

Producers are modelled as firms that belong to sectors. Here, both the notions of firm and sector are versatile: a single firm with subsidiaries in different countries and/or offering differentiated products can be modelled as a set of independent firms. Similarly, a sector refers to a group of firms with similar activities, but this criteria is loose: sectors can be decomposed in sector per country, to account for local specificities, or in several sectors for that matter.

Producers move across sectors described by a vector space of arbitrary dimension. The position of producer i in this space is denoted X_i and his physical capital, K_i. Producers are defined by these two variables, which are both subject to dynamic changes. Producers may change their capital stocks over time, or altogether shift sector.

Each firm produces a single differentiated good. However in the following we will merely consider the return each producer may provide to its investors.

The return of producer i at time t, denoted r_i, depends on K_i, X_i and on the level of competition in the sector. It is written:

$$r_i = r\left(K_i, X_i\right) - \gamma \sum_j \delta\left(X_i - X_j\right) \frac{K_j}{K_i} \tag{1}$$

The first term is an arbitrary function that depends on the sector and the level of capital per firm in this sector. It represents the return of capital in a specific sector X_i under no competition. We deliberately keep the form of $r\left(K_i, X_i\right)$ unspecified, since most of the results of the model rely on general properties of the functions involved. When needed, we will give a standard Cobb-Douglas form to the returns $r\left(K_i, X_i\right)$.

The second term in (1) is the decreasing return of capital. In any given sector, it is proportional to both the number of competitors and the specific level of capital per firm used.

We also assume that, for all i, firm i has a market valuation defined by both its price P_i and the variation $\dot{P}_i$ of this price on financial markets. This variation is assumed to be a function of an expected long-term return denoted $R(K_i, X_i)$, such that:

$$\frac{\dot{P}_i}{P_i} = F_1 \left(\frac{R(K_i, X_i)}{\sum_l R(K_l, X_l)} \right) \tag{2}$$

where the quantity $\frac{R(K_i, X_i)}{\sum_l R(K_l, X_l)}$ is the relative return of firm i relative to the whole set of firms in its sector. The function F_1 is arbitrary and reflects the preferences of the market relatively to the firm's relative returns.

We assume that firms shift their production in the sector space according to returns, in the direction of the gradient of the expected long-term return $R(K_i, X_i)$. Yet, the accumulation of agents at any point of the space creates a repulsive force, so that the evolution of X_i minimizes, up to some shocks, the following quantity:

$$L_i \left(X_i, \frac{dX_i}{dt} \right) = \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 + \tau \sum_j \delta(X_i - X_j) \tag{3}$$

When $\tau = 0$, there are no repulsive forces and the move towards the gradient of R is given by the expression:

$$\frac{dX_i}{dt} = \nabla_X R(K_i, X_i) H(K_i)$$

When $\tau \neq 0$, repulsive forces deviate the trajectory. The dynamic equation associated to the minimization of (3) is given by the general formula of the dynamic optimization:

$$\frac{d}{dt} \frac{\partial}{\partial \frac{dX_i}{dt}} L_i \left(X_i, \frac{dX_i}{dt} \right) = \frac{\partial}{\partial X_i} L_i \left(X_i, \frac{dX_i}{dt} \right) \tag{4}$$

This last equation does not need to be developed further, since formula (3) is sufficient to switch to the field description of the system. Note for later purpose that the expression $\frac{dX_i}{dt}$ stands for the continuous version of a discrete variation, $X_i(t+1) - X_i(t)$.

3.2   Investors

Each investor j is defined by his level of capital $\hat{K}_j$ and his position $\hat{X}_j$ in the sector space. Investors can invest in the entire sector space, but tend to invest in sectors close to their position.

Besides, investors tend to diversify their capital: each investor j chose to allocate parts of his entire capital $\hat{K}_j$ between various firms i. The capital allocated by investor j to firm i is denoted $\hat{K}_j^{(i)}$, and given by:

$$\hat{K}_j^{(i)}(t) = \left(\frac{F_2(R(K_i, X_i)) G(X_i - \hat{X}_j)}{\sum_l F_2(R(K_l, X_l)) G(X_l - \hat{X}_j)} \hat{K}_j \right)(t) \tag{5}$$

where F_2 is an arbitrary function that depends on the expected return of firm i and the distance between sector X_i et $\hat{X}_j$.

We define ε the time scale for capital accumulation. The variation of capital of investor j between t and $t+\varepsilon$ is the sum of two terms: the short-term returns r_i of the firms in which j invested, and the stock price variations of these same firms:

$$\hat{K}_j(t+\varepsilon) - \hat{K}_j(t) = \sum_i \left(r_i + \frac{\dot{P}_i}{P_i} \right) \hat{K}_j^{(i)} = \sum_i \left(r_i + F_1 \left(\frac{R(K_i, X_i)}{\sum_l R(K_l, X_l)}, \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \hat{K}_j^{(i)} \tag{6}$$

Incidentally, note that in equation (3), the time scale of motions within the sectors space was normalized to one. Here, on the contrary, we define this motion time scale as ε, and assume $\varepsilon << 1$: the mobility in the sector space is lower than capital dynamics. To rewrite (6) on the same time-span as $\frac{dX_i}{dt}$, we write:

$$\hat{K}_j\left(t+1\right)-\hat{K}_j\left(t\right) \quad = \quad \sum_{k=1}^{\frac{1}{\varepsilon}}\hat{K}_j\left(t+k\varepsilon\right)-\hat{K}_j\left(t\right)$$

$$= \quad \sum_{k=1}^{\frac{1}{\varepsilon}}\sum_i\left(r_i+\frac{\dot{P}_i}{P_i}\right)\hat{K}_j^{(i)}\left(t+k\varepsilon\right)\simeq\frac{1}{\varepsilon}\sum_i\left(r_i+F_1\left(\frac{R\left(K_i,X_i\right)}{\sum_l R\left(K_l,X_l\right)},\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\hat{K}_j^{(i)}$$

where the quantities in the sum have to be understood as averages over the time span $[t,t+1]$. Using equation (2), equation (6) becomes in the continuous approximation:

$$\frac{d}{dt}\hat{K}_j\left(t\right)=\frac{1}{\varepsilon}\sum_i\left(r_i+F_1\left(\frac{R\left(K_i,X_i\right)}{\sum_l\delta\left(X_l-X_i\right)R\left(K_l,X_l\right)},\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\frac{F_2\left(R\left(K_i,X_i\right)\right)G\left(X_i-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l,X_l\right)\right)G\left(X_l-\hat{X}_j\right)}\hat{K}_j \quad (7)$$

where $\frac{d}{dt}\hat{K}_j\left(t\right)=\hat{K}_j\left(t+1\right)-\hat{K}_j\left(t\right)$ is now normalized to the time scale of $\frac{dX_i}{dt}$, i.e. 1.

3.3 Link between financial and physical capital

The entire financial capital is, at any time, completely allocated by investors between firms. There is no alternative source of financing for producers: self-financing is discarded, since it essentially amounts to consider two agents, a producer and an investor, as one. The physical capital of a any given firm is thus the sum of all capital allocated to this firm by all its investors. Physical capital entirely depends on the arbitrage and allocations of the financial sector. Firms do not own their capital: they fully return it at the end of each period with a dividend, though possibly negative. Investors then chose to reallocate their funds in their entirety between firms at the beginning of the next period.

This set up is a generalisation of the dividend irrelevance theory. It may not be fully accurate in the short-run but, since physical capital cannot subsist without investment, it holds in the long-run. When investors choose not to finance a firm, this firm is bound to disappear in the long run. Under these assumptions, the following identity holds:

$$K_i\left(t+\varepsilon\right)=\sum_j\hat{K}_j^{(i)}=\sum_j\frac{F_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right)\right)G\left(X_i\left(t\right)-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right),X_l\left(t\right)\right)\right)G\left(X_l\left(t\right)-\hat{X}_j\right)}\hat{K}_j\left(t\right) \quad (8)$$

where K_i stands for the physical capital of firm i at time t, and $\sum_j\hat{K}_j^{(i)}$ for the sum of capital invested in firm i by investors j. Recall that the parameter ε accounts for the specific time scale of capital accumulation. It differs from that of mobility within the sector space (3), which is normalized to one.

The dynamics (8) rewrites:

$$\frac{K_i\left(t+\varepsilon\right)-K_i\left(t\right)}{\varepsilon}=\frac{1}{\varepsilon}\left(\sum_j\frac{F_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right)\right)G\left(X_i\left(t\right)-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right),X_l\left(t\right)\right)\right)G\left(X_l\left(t\right)-\hat{X}_j\right)}\hat{K}_j\left(t\right)-K_i\left(t\right)\right) \quad (9)$$

Using the same token as in the derivation of (7), we obtain in the continuous approximation:

$$\frac{d}{dt}K_i\left(t\right)+\frac{1}{\varepsilon}\left(K_i\left(t\right)-\sum_j\frac{F_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right)\right)G\left(X_i\left(t\right)-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right),X_l\left(t\right)\right)\right)G\left(X_l\left(t\right)-\hat{X}_j\right)}\hat{K}_j\left(t\right)\right)=0 \quad (10)$$

where $\frac{d}{dt}K_i\left(t\right)$ stands for $K_i\left(t+1\right)-K_i\left(t\right)$.

3.4 Capital allocation dynamics

Investors choose to allocate their capital within sectors, and may modify their portfolio according to the returns of the sector or firms they invest in. This is modelled by a move along the sectors' space in the direction of the gradient of $R(K_i, X_i)$. The move of $\hat{X}_j$ is described by the dynamic equation:

$$\frac{d}{dt}\hat{X}_j - \frac{1}{\sum_i \delta\left(X_i - \hat{X}_j\right)} \sum_i \left(\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\frac{R\left(K_i, \hat{X}_j\right)}{\sum_l R(K_l, X_l)}\right)\right) = 0 \qquad (11)$$

where the factor $\sum_i \delta\left(X_i - \hat{X}_j\right)$ is the agents' density in the sector $\hat{X}_j$, so that the more competitors in a sector, the slower the move.

In equation (??), the term $\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right)$ is the tendency of investors to invest in sectors with the highest returns, which can be described by a move in the direction defined by a function F_0 of long-term returns.

The term $\nu \nabla_{\hat{X}} F_1\left(\frac{R\left(K_i, \hat{X}_j\right)}{\sum_l R(K_l, X_l)}\right)$ describes the investors' preference for stocks with the highest price-dividend ratio.

Ultimately, note that unlike K and $\hat{K}$, X and $\hat{X}$ are not a strictly standard economic variables, and that their dynamics should thus be ascribed an ad-hoc form.

4 Fields formalism

In the above, we have detailed a standard, classical microeconomic framework. We will now present a general method to translate such a framework into a field model.

4.1 General Method

To transform our framework into a field model, we must first consider the types of agents in the model, and rewrite their dynamics as the minimization equations of some initial functions, in the same way as, for instance, consumption dynamics could be derived from an utility function.

Since each type of agent in our framework is described by two dynamic equations, there are four minimization functions to find. These minimization functions will be translated into four functionals of two independent fields[3], one for producers, the real economy, and one for investors, financial markets. The sum of the four functionals is the "field action functional" that describes the whole system in terms of fields[4].

4.1.1 Minimization functions

In standard economic frameworks, each type of agent is characterized by one or more dynamic equations. Some of these dynamic equations can result from a minimization, while others do not. The translation into fields is built upon the functions which, once minimized, yields the system's dynamics. Two cases arise.

When the dynamics are constructed from minimizations, it suffices to use the function which, minimized, gave the equation of the dynamics. These functions, related to the probabilistic interpretation, represent the deviation between the trajectory of an agent and an average or optimal trajectory. They are therefore directly linked to the number of agents in the system.

However, dynamics may not always result from a minimization. In this second case, we must ad-hoc reconstruct functions whose minimization would restore the dynamic equations. Such functions refer to the probabilistic interpretation of the system. They are not unique. When modelling heterogeneous agents, quadratic functions allow to translate the quadratic deviation from the mean trajectory of agents subject to idiosyncratic shocks. This quadratic deviation represents the variance of these shocks and is directly related to the probability of deviating from an average trajectory (see GLW). The construction of these quadratic functions is straightforward.

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[3] The term functional refers to a function of a function, i.e. a function whose argument is itself a function.

[4] Details about the probabilistic step will be given as a reminder along the text and in appendix 1.

In general, we assume agents are described by vectors $\mathbf{A}_i(t)$ of arbitrary dimension, where $\mathbf{A}_i(t)$ satisfies a dynamic equation characterizing agent i:

$$\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right) = 0 \tag{12}$$

This type of equation, which involves the whole set of other agents, is characteristic of models with a large number of interacting agents. Its associated minimization function is obtained directly, by first squaring the lhs of (12), and then summing over the whole set of agents to describe the full system:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{13}$$

However, some minimization functions may also include an additional term, without any time derivative (see for example equation (3)), so that the most general minimization function writes, for some function g:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\right)^2 + \sum_i \sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right) \tag{14}$$

Ultimately, the sum s of all the minimization functions is the sum of all agents' squared deviation from the average dynamics within the system[5]. A generalisation of equation (14), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

4.1.2 Translation in terms of fields

The translation itself can be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

Indexed variables, without temporal derivative The terms in (14) that include indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)$$

These terms describe the whole set of interactions both among and between two groups of agents. Here, agents are characterized by their variables $\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t)...$ and $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ respectively, for instance in our model firms and investors.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively.

In a first step, the variables indexed i such as $\mathbf{A}_i(t)$ are replaced by variables $\mathbf{A}$ in the expression of g. The variables indexed $j,k,l,m...$, such as $\mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ are replaced by $\mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'$, and so on for all the indices in the function. This yields the expression:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)$$

In a second step, each sum is replaced by a weighted integration symbol:

$$\sum_i \rightarrow \int |\Psi(\mathbf{A})|^2 d\mathbf{A}, \sum_j \rightarrow \int |\Psi(\mathbf{A}')|^2 d\mathbf{A}', \sum_k \rightarrow \int |\Psi(\mathbf{A}'')|^2 d\mathbf{A}''$$

$$\sum_l \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}}, \sum_m \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}'$$

---

[5] The function s is related to the probabilistic approach, which associates the probability $\exp(-s)$ to the state of the system defined by the $A_i(t), A_j(t),..$ (see appendix 1)

which leads to the translation:

$$\sum_i \sum_j \sum_{j,k...} g\left(\mathbf{A}_i\left(t\right), \mathbf{A}_j\left(t\right), \mathbf{A}_k\left(t\right), \hat{\mathbf{A}}_l\left(t\right), \hat{\mathbf{A}}_m\left(t\right)...\right)$$

$$\rightarrow \int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) \left|\Psi\left(\mathbf{A}\right)\right|^2 \left|\Psi\left(\mathbf{A}'\right)\right|^2 \left|\Psi\left(\mathbf{A}''\right)\right|^2 d\mathbf{A} d\mathbf{A}' d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \quad (15)$$

Variable with temporal derivative The terms in (14) that imply a variable temporal derivative are of the form:

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}\left(t\right)}{dt} - \sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i\left(t\right), \mathbf{A}_j\left(t\right), \mathbf{A}_k\left(t\right), \hat{\mathbf{A}}_l\left(t\right), \hat{\mathbf{A}}_m\left(t\right)...\right)\right)^2 \quad (16)$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i\left(t\right)$ as a function of the other agents.

The method of translation is similar to the above, but the time derivative adds an additional operation. In a first step, we translate the terms without derivative inside the parenthesis:

$$\sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i\left(t\right), \mathbf{A}_j\left(t\right), \mathbf{A}_k\left(t\right), \hat{\mathbf{A}}_l\left(t\right), \hat{\mathbf{A}}_m\left(t\right)...\right) \quad (17)$$

This type of term has already been translated in the previous paragraph, but since there is no sum over i in (17), there should be no integral over $\mathbf{A}$, nor factor $\left|\Psi\left(\mathbf{A}\right)\right|^2$.

The translation of (17) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) \left|\Psi\left(\mathbf{A}'\right)\right|^2 \left|\Psi\left(\mathbf{A}''\right)\right|^2 d\mathbf{A}' d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \quad (18)$$

A free variable $\mathbf{A}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{A})$ the expression obtained:

$$\Lambda(\mathbf{A}) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) \left|\Psi\left(\mathbf{A}'\right)\right|^2 \left|\Psi\left(\mathbf{A}''\right)\right|^2 d\mathbf{A}' d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \quad (19)$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}\left(t\right)}{dt} - \sum_j \sum_{j,k...} f^{(\alpha)}\left(\mathbf{A}_i\left(t\right), \mathbf{A}_j\left(t\right), \mathbf{A}_k\left(t\right), \hat{\mathbf{A}}_l\left(t\right), \hat{\mathbf{A}}_m\left(t\right)...\right)\right)^2 \quad (20)$$

by:

$$\int \Psi^{\dagger}\left(\mathbf{A}\right) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma_{\mathbf{A}^{(\alpha)}}^2}{2} \nabla_{\mathbf{A}^{(\alpha)}} + \Lambda(\mathbf{A})\right)\right) \Psi\left(\mathbf{A}\right) d\mathbf{A} \quad (21)$$

The variance $\sigma_{\mathbf{A}^{(\alpha)}}^2$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (21), the integral over $\mathbf{A}$ reappears at the end, along with the square of the field $\left|\Psi\left(\mathbf{A}\right)\right|^2$. This square is split into two terms, $\Psi^{\dagger}\left(\mathbf{A}\right)$ and $\Psi\left(\mathbf{A}\right)$, with a gradient operator inserted in between.

4.1.3 Gathering terms

The field description is ultimately obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional. It is the sum of terms of the form (15) and (21), and is denoted $S\left(\Psi, \Psi^{\dagger}\right)$.

Introducing time in the model Until now, no time variable was included in this model. We now introduce one, written θ to distinguish it from the classical model variables. We replace:

$$\Psi\left(\mathbf{A}\right) \;\rightarrow\; \Psi\left(\mathbf{A},\theta\right)$$
$$\hat{\Psi}\left(\hat{\mathbf{A}}\right) \;\rightarrow\; \hat{\Psi}\left(\hat{\mathbf{A}},\theta\right)$$

and introduce an additional contribution to $S\left(\Psi,\Psi^{\dagger}\right)$ (see GLW). The full action functional then becomes:

$$S\left(\Psi,\Psi^{\dagger}\right) + \Psi^{\dagger}\left(\mathbf{A},\theta\right)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-1\right)\right)\Psi\left(\mathbf{A},\theta\right) \tag{22}$$
$$+\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}},\theta\right)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-1\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}},\theta\right) + \alpha\left|\Psi\left(\mathbf{A}\right)\right|^{2} + \alpha\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}$$

where σ_{θ}^{2} is a variance term accounting for delays in interactions, and $\frac{1}{\alpha}$ is a time scale describing the average time span of interactions between agents. In practice $\sigma_{\theta}^{2} << 1$ and $\alpha << 1$.

Remark about the introduction of the time variable As mentioned above, equation (22) is necessary to describe some time-dependent processes.

However, including this time-variable in the model is not always necessary. In this paper for instance, we are solely interested in the background fields of the system. The background fields refer to a long-run, stationary-type of equilibrium. As such, we can avoid introducing a time variable in the model, and simply consider static fields, $\Psi\left(\mathbf{A}\right)$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$.

We will later on include some time-dependent modifications in our background fields. However, these modifications will reflect modifications in the parameters, that must not be confused with the short-term θ dependency.

5 Application to the framework

5.1 Minimization functions

In our model, the dynamics of the variable X_i comes from the minimization of the function:

$$\left(\frac{dX_i}{dt} - \nabla_X R\left(K_i,X_i\right)H\left(K_i\right)\right)^2 + \tau\sum_{j}\delta\left(X_i-X_j\right)$$

We simply re-use this function. Since we are interested in the whole system, we will sum over the whole set of agents, which yields the minimization function for the capital allocation dynamics:

$$\sum_{i}\left(\frac{dX_i}{dt} - \nabla_X R\left(K_i,X_i\right)H\left(K_i\right)\right)^2 + \sum_{i}\tau\sum_{j}\delta\left(X_i-X_j\right) \tag{23}$$

On the contrary, the dynamics of the variables K_i, $\hat{K}_i$ et $\hat{X}_i$ are not the result of a minimization, but their associated quadratic functions (13) can easily be found. These functions are therefore:

for K_i, the minimization function for physical capital dynamics:

$$\sum_{i}\left(\frac{d}{dt}K_i + \frac{1}{\varepsilon}\left(K_i - \sum_{j}\frac{F_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right)\right)G\left(X_i\left(t\right)-\hat{X}_j\right)}{\sum_{l}F_2\left(R\left(K_l\left(t\right),X_l\left(t\right)\right)\right)G\left(X_l\left(t\right)-\hat{X}_j\right)}\hat{K}_j\left(t\right)\right)\right)^2 \tag{24}$$

for $\hat{K}_i$, the minimization function for the financial capital dynamics:

$$\sum_{j}\left(\frac{d}{dt}\hat{K}_j - \frac{1}{\varepsilon}\left(\sum_{i}\left(r_i + F_1\left(\frac{R\left(K_i,X_i\right)}{\sum_{l}\delta\left(X_l-X_i\right)R\left(K_l,X_l\right)},\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\frac{F_2\left(R\left(K_i,X_i\right)\right)G\left(X_i-\hat{X}_j\right)}{\sum_{l}F_2\left(R\left(K_l,X_l\right)\right)G\left(X_l-\hat{X}_j\right)}\hat{K}_j\right)\right)^2 \tag{25}$$

and for $\hat{X}_i$, the minimization function for financial capital allocation:

$$\sum_i \left(\frac{d}{dt}\hat{X}_j - \frac{1}{\sum_i \delta\left(X_i - \hat{X}_j\right)} \sum_i \left(\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\frac{R\left(K_i, \hat{X}_j\right)}{\sum_l R\left(K_l, X_l\right)} \right) \right) \right)^2 \tag{26}$$

5.2 Translation in terms of fields

We apply the general method developed above and translate the minimization functions (23), (24), (25) and (26) in terms of fields. We start with producers, and translate first (23) and (24).

5.2.1 The Real Economy

In both capital allocation dynamics (23) and capital accumulation dynamics (24), time derivatives appear. However, one of them, equation (23), includes time-independent terms and is thus of the form (14), the other, equation (23) is of the type (13).

Translation of the minimization function: Physical capital allocation Let us start by translating in terms of fields the expression (23):

$$\sum_i \left(\left(\frac{dX_i}{dt} - \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)\right)^2 + \tau \sum_j \delta\left(X_i - X_j\right) \right) \tag{27}$$

To do so, we first consider the last term $\tau \sum_i \sum_j \delta\left(X_i - X_j\right)$. This term contains no derivative. The form of the translation is given by formula (15). Since the expression contains two indices, both of them are summed.

The first step of the translation is to replace X_i and X_j by two variables X et X', and substitute:

$$\tau\delta\left(X_i - X_j\right) \to \tau\delta\left(X - X'\right)$$

The sum over i and the sum over j are then replaced directly by the integrals $\int |\Psi\left(K, X\right)|^2 d\left(K, X\right)$, $\int |\Psi\left(K', X'\right)|^2 d\left(K', X'\right)$, which leads to the following translation:

$$\begin{aligned}
\tau \sum_i \sum_j \delta\left(X_i - X_j\right) &\to \int |\Psi\left(K, X\right)|^2 d\left(K, X\right) \int |\Psi\left(K', X'\right)|^2 d\left(K', X'\right) \tau\delta\left(X - X'\right) \\
&= \int \tau |\Psi\left(K, X\right)|^2 |\Psi\left(K', X\right)|^2 d\left(K, X\right) dK'
\end{aligned} \tag{28}$$

To translate the first term in formula (27):

$$\sum_i \left(\frac{dX_i}{dt} - \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)\right)^2 \tag{29}$$

We use the translation (21) of a type-(20) expression. The gradient term appearing in equation (21) is ∇_X. We thus obtain the translation:

$$\begin{aligned}
&\sum_i \left(\frac{dX_i}{dt} - \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)\right)^2 \tag{30} \\
&\to \int \Psi^\dagger\left(K, X\right) \left(-\nabla_X \left(\frac{\sigma_X^2}{2}\nabla_X + \Lambda\left(X, K\right)\right) \right) \Psi\left(K, X\right) dK dX
\end{aligned}$$

Note that the variance σ_X^2 reflects the probabilistic nature of the model hidden behind the field formalism. This σ_X^2 represents the characteristic level of uncertainty of the sectors space dynamics. It is a parameter

of the model. The term $\Lambda(X, K)$ is the translation of the term $-\nabla_X R\left(K_i, X_i\right) H\left(K_i\right)$ in the parenthesis of (29). This term is a function of one sole index "i". In that case, the term Λ is simply obtained by replacing $\left(K_i, X_i\right)$ by (K, X). We use the translation (19) of (17)-type term, so that Λ writes:

$$\Lambda(X, K) = -\nabla_X R\left(K, X\right) H\left(K\right)$$

and the translation of expression (29) is:

$$\sum_i \left(\frac{dX_i}{dt} - \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)\right)^2 \tag{31}$$

$$\to \quad \int \Psi^\dagger\left(K, X\right)\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K, X\right) H\left(K\right)\right)\right)\Psi\left(K, X\right) dK dX$$

Using equations (28) and (31), the translation of (27) is thus:

$$S_1 \quad = \quad -\int \Psi^\dagger\left(K, X\right)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K, X\right) H\left(K\right)\right)\Psi\left(K, X\right) dK dX \tag{32}$$

$$+\tau \int \left|\Psi\left(K', X\right)\right|^2 \left|\Psi\left(K, X\right)\right|^2 dK' dK dX$$

Translation of the minimization function: Physical capital We can now turn to the translation of the second equation (24), which we rewrite for the sake of clarity as:

$$\sum_i \left(\frac{d}{dt}K_i + \frac{1}{\varepsilon}\left(K_i\left(t\right) - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\right)}\hat{K}_j\left(t\right)\right)\right)^2 \tag{33}$$

Once again, we use the translation (19) of (17)-type term, and start by building the field functional associated to the term inside the square:

$$K_i\left(t\right) - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\left(t\right)\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\left(t\right)\right)}\hat{K}_j\left(t\right)$$

We replace:

$$\begin{aligned} \left(K_i\left(t\right), X_i\left(t\right)\right) &\to \left(K, X\right) \\ \left(K_l\left(t\right), X_l\left(t\right)\right) &\to \left(K', X'\right) \\ \left(\hat{K}_j\left(t\right), \hat{X}_j\left(t\right)\right) &\to \left(\hat{K}, \hat{X}\right) \end{aligned}$$

and:

$$K_i\left(t\right) - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\right)}\hat{K}_j\left(t\right) \to K - \sum_j \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right)}{\sum_l F_2\left(R\left(K', X'\right)\right) G\left(X' - \hat{X}\right)}\hat{K} \tag{34}$$

The sum over l is then replaced by an integral $\int \left|\Psi\left(K', X'\right)\right|^2 d\left(K', X'\right)$:

$$K_i\left(t\right) - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\right)}\hat{K}_j\left(t\right) \tag{35}$$

$$\to \quad K - \sum_j \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right)}{\int \left|\Psi\left(K', X'\right)\right|^2 d\left(K', X'\right) F_2\left(R\left(K', X'\right)\right) G\left(X' - \hat{X}_j\right)}\hat{K}$$

13

Recall that investors' variables are denoted with an upper script ˆ.

Finally, the sum over j and the second field are replaced by $\int \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 d\left(\hat{K}, \hat{X}\right)$. After introducing the characteristic factor $\frac{1}{\varepsilon}$ of the capital accumulation time scale (see (9)), the translation becomes:

$$
\frac{1}{\varepsilon}\left(K_i\left(t\right) - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\right)} \hat{K}_j\left(t\right)\right)
$$

$$
\rightarrow \quad \frac{1}{\varepsilon}\left(K - \int \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 d\left(\hat{K}, \hat{X}\right) \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right) \hat{K}}{\int \left|\Psi\left(K', X'\right)\right|^2 d\left(K', X'\right) F_2\left(R\left(K', X'\right)\right) G\left(X' - \hat{X}\right)}\right)
$$

$$
\equiv \quad \Lambda\left(K, X\right) \tag{36}
$$

Using the translation (21) of (20)-type term, we are led to the translation of (33). Since the square (33) includes a derivative $\frac{d}{dt}K_i$, the expression starts with a gradient with respect to K, and we have:

$$
\sum_i \left(\frac{d}{dt}K_i + \frac{1}{\varepsilon}\left(K_i - \sum_j \frac{F_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right)\right) G\left(X_i\left(t\right) - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l\left(t\right), X_l\left(t\right)\right)\right) G\left(X_l\left(t\right) - \hat{X}_j\right)} \hat{K}_j\left(t\right)\right)\right)^2 \tag{37}
$$

$$
\rightarrow \quad \int \Psi^\dagger\left(K, X\right)\left(-\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + \Lambda\left(K, X\right)\right)\right)\Psi\left(K, X\right) dK dX
$$

where, here again, the variance σ_K^2 reflects the probabilistic nature of the model that is hidden behind the field formalism. Recall that it represents the characteristic level of uncertainty in the dynamics of capital.

Inserting result (36) in equation (37), the translation of (**??**) becomes:

$$
S_2 = -\int \Psi^\dagger\left(K, X\right) \nabla_K \left(\frac{\sigma_K^2}{2}\nabla_K + \frac{1}{\varepsilon}\left(K - \int \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right) \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X}}{\int F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right) \left\|\Psi\left(K, X\right)\right\|^2}\right)\right) \Psi\left(K, X\right) \tag{38}
$$

5.2.2   Financial markets

The functions to be translated are those of the financial capital dynamics (25) and of the financial capital allocation (26). Both expressions include a time derivative and are thus of type (16). As for the real economy, the application of the translation rules is straightforward.

Translation of the minimization function: Financial capital dynamics   We consider the function (25):

$$
\sum_j \left(\frac{d}{dt}\hat{K}_j - \frac{1}{\varepsilon}\left(\sum_i \left(r_i + F_1\left(\frac{R\left(K_i, X_i\right)}{\sum_l \delta\left(X_l - X_i\right) R\left(K_l, X_l\right)}, \frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right) \frac{F_2\left(R\left(K_i, X_i\right)\right) G\left(X_i - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l, X_l\right)\right) G\left(X_l - \hat{X}_j\right)} \hat{K}_j\right)\right)^2 \tag{39}
$$

which translates, using the general translation formula of expression (20) in (21), into:

$$
\int \hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)\left(-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} + \Lambda\left(\hat{K}, \hat{X}\right)\right)\right)\hat{\Psi}\left(\hat{K}, \hat{X}\right) d\hat{K} d\hat{X}
$$

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The function $\Lambda\left(\hat{K},\hat{X}\right)$ is obtained, as before, by translating the term following the derivative in the function (39):

$$\frac{1}{\varepsilon}\sum_i\left(r_i+F_1\left(\frac{R\left(K_i,X_i\right)}{\sum_l\delta\left(X_l-X_i\right)R\left(K_l,X_l\right)},\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\frac{F_2\left(R\left(K_i,X_i\right)\right)G\left(X_i-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l,X_l\right)\right)G\left(X_l-\hat{X}_j\right)}\hat{K}_j\rightarrow\Lambda\left(\hat{K},\hat{X}\right)$$

(40)

First, we use the price dynamics equation (2) at the zero-th order in fluctuations to translate the capital dynamics $\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}$:

$$
\begin{aligned}
\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)} &= \sum_j\frac{F_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right)\right)G\left(X_i\left(t\right)-\hat{X}_j\right)}{K_i\sum_l F_2\left(R\left(K_l\left(t\right),X_l\left(t\right)\right)\right)G\left(X_l\left(t\right)-\hat{X}_j\right)}\hat{K}_j\left(t\right)-K_i\left(t\right)\\
&\rightarrow \Gamma\left(K,X\right)
\end{aligned}
$$

where:

$$
\begin{aligned}
\Gamma\left(K,X\right) &= \frac{\int\frac{F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)\|\Psi\left(K,X\right)\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right)-K}{K}\\
&= \int\frac{F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{K\int F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)\|\Psi\left(K,X\right)\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right)-1
\end{aligned}
$$

(41)

Then, using the translation (19) of (17), we translate expression (40) by replacing:

$$
\begin{aligned}
\left(K_i,X_i\right) &\rightarrow \left(K,X\right)\\
\left(K_l,X_l\right) &\rightarrow \left(K',X'\right)\\
\left(\hat{K}_j,\hat{X}_j\right) &\rightarrow \left(\hat{K},\hat{X}\right)
\end{aligned}
$$

We also replace the sums by integrals times the appropriate square of field, which yields:

$$
\begin{aligned}
\Lambda\left(\hat{K},\hat{X}\right) &= -\frac{\hat{K}}{\varepsilon}\int\left(r\left(K,X\right)-\gamma\frac{\int K'\|\Psi\left(K',X\right)\|^2}{K}+F_1\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&\quad\times\frac{F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_2\left(R\left(K',X'\right)\right)G\left(X'-\hat{X}\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)}\|\Psi\left(K,X\right)\|^2 d\left(K,X\right)
\end{aligned}
$$

Ultimately, the translation of (25) is:

$$
\begin{aligned}
S_3 &= -\int\hat{\Psi}^\dagger\left(\hat{K},\hat{X}\right)\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\frac{\hat{K}}{\varepsilon}\int\left(r\left(K,X\right)-\gamma\frac{\int K'\|\Psi\left(K',X\right)\|^2}{K}\right.\right.\\
&\quad\left.+F_1\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&\quad\left.\times\frac{F_2\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_2\left(R\left(K',X'\right)\right)G\left(X'-\hat{X}\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)}\|\Psi\left(K,X\right)\|^2 d\left(K,X\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)
\end{aligned}
$$

Translation of the minimization function: Financial capital allocation   The translation of the function for financial capital allocation (26) follows the previous pattern. We obtain:

$$
S_4 = -\int \hat{\Psi}^\dagger \left(\hat{K}, \hat{X}\right) \nabla_{\hat{X}} \left(\sigma_{\hat{X}}^2 \nabla_{\hat{X}} - \int \left(\nabla_{\hat{X}} F_0 \left(R\left(K, \hat{X}\right)\right) + \nu \nabla_{\hat{X}} F_1 \left(\frac{R\left(K, \hat{X}\right)}{\int R\left(K', X'\right) \left\| \Psi\left(K', X'\right)\right\|^2 d\left(K', X'\right)}\right)\right)\right.
$$

$$
\times \left. \frac{\left\| \Psi\left(K, \hat{X}\right)\right\|^2 dK}{\int \left\| \Psi\left(K', \hat{X}\right)\right\|^2 dK'}\right) \hat{\Psi}\left(\hat{K}, \hat{X}\right)
$$

5.3   Gathering contributions: the action functional

Once these translations are performed, the action functional of the system is described by the sum of all contributions:

$$
S = S_1 + S_2 + S_3 + S_4
$$

We write a compact form for the action functional S:

$$
S = -\int \Psi^\dagger (K, X) \left(\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K)\right) - \tau \left(\int \left| \Psi\left(K', X\right)\right|^2 dK'\right)\right. \tag{42}
$$

$$
\left. + \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u\left(K, X, \Psi, \hat{\Psi}\right)\right)\right) \Psi(K, X)\, dK dX
$$

$$
- \int \hat{\Psi}^\dagger \left(\hat{K}, \hat{X}\right) \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f\left(\hat{X}, \Psi, \hat{\Psi}\right)\right) + \nabla_{\hat{X}} \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}} - g\left(K, X, \Psi, \hat{\Psi}\right)\right)\right) \hat{\Psi}\left(\hat{K}, \hat{X}\right)
$$

where each line corresponds to one S_i and where, to simplify, we have defined:

$$
u\left(K, X, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon} \left(K - \int \frac{F_2(R(K, X)) G\left(X - \hat{X}\right)}{\int F_2(R(K, X)) G\left(X - \hat{X}\right) \left\| \Psi(K, X)\right\|^2} \hat{K} \left\| \hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X}\right) \tag{43}
$$

$$
f\left(\hat{X}, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon} \int \left(r(K, X) - \frac{\gamma \int K' \left\| \Psi(K, X)\right\|^2}{K} + F_1 \left(\frac{R(K, X)}{\int R\left(K', X'\right) \left| \Psi\left(K', X'\right)\right|^2 dK' dX'}, \Gamma(K, X)\right)\right)
$$

$$
\times \frac{F_2(R(K, X)) G\left(X - \hat{X}\right)}{\int F_2\left(R\left(K', X'\right)\right) G\left(X' - \hat{X}\right) \left\| \Psi\left(K', X'\right)\right\|^2 d\left(K', X'\right)} \left| \Psi(K, X)\right|^2 d(K, X) \tag{44}
$$

$$
g\left(K, \hat{X}, \Psi, \hat{\Psi}\right) = \int \left(\nabla_{\hat{X}} F_0 \left(R\left(K, \hat{X}\right)\right)\right. \tag{45}
$$

$$
\left. + \nu \nabla_{\hat{X}} F_1 \left(\frac{R\left(K, \hat{X}\right)}{\int R\left(K', X'\right) \left\| \Psi\left(K', X'\right)\right\|^2 d\left(K', X'\right)}, \Gamma(K, X)\right)\right) \frac{\left\| \Psi\left(K, \hat{X}\right)\right\|^2 dK}{\int \left\| \Psi\left(K', \hat{X}\right)\right\|^2 dK'}
$$

The expression of $\Gamma(K, X)$ has been given in (41).

Recall that function $H(K_X)$ encompasses the determinants of the firms' mobility across the sector space. We will specify this function below as a function of expected long term-returns and capital.

Function u describes the evolution of capital of a firm, located at X. This dynamics depends on the relative value of a function F_2 that is itself a function of the firms' expected returns $R(K, X)$. Investors allocate their capital based on their expectations of the firms' long-term returns.

Function f describes the returns of investors located at $\hat{X}$, and investing in sector X a capital K. These returns depend on short-term dividends $r(K, X)$, the field-equivalent cost of capital $\frac{\gamma \int K' \| \Psi(K, X)\|^2}{K}$, and a function F_1 that depends on firms' expected long-term stock valuations. These valuations themselves depend on the relative attractivity of a firm expected long-term returns vis-a-vis its competitors.

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Function g describes investors' shifts across the sectors' space. They are driven by the gradient of expected long-term returns and stocks valuations, who themselves depend on the firms' relative expected long-term returns.

Recall that we depart here from the general formalism: we do not introduce a time variable in the present model. Indeed, as mentioned earlier, our purpose is to find collective, or characteristic, configurations of the system that, as such, can be considered static. It is only when we will derive these configurations that a macro time scale will be introduced to study how the evolution of the background states through time.

6 Use of the field model

Now that we have found the field action functional S, we can use field theory to study the system of agents. This can be done at two levels.

At the collective level At the collective level, the background fields of the system, i.e. the particular functions $\Psi(K, X)$ and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ that minimize the functional S, can be computed. The functions, squared, $|\Psi(K, X)|^2$ and $\left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2$, represent the density of agents, both per sector and for a given capital K, in the collective state defined by $\Psi(K, X)$ and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$. Thus, the collective state determines, for each sector and for a given capital K, the density of firms and the density of investors.

Moreover, these two squared functions allow to compute various global quantities of the system in the collective state $\Psi(K, X)$ and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$.

The sectors' number of producers $N(X)$ and investors $\hat{N}\left(\hat{X}\right)$ are computed using the formula:

$$
N(X) = \int |\Psi(K, X)|^2 \, dK \tag{46}
$$

$$
\hat{N}\left(\hat{X}\right) = \int \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 \, d\hat{K} \tag{47}
$$

The average values of total invested capital $\hat{K}_X$ for each sector X is:

$$
\hat{K}_{\hat{X}} = \int \hat{K} \left|\hat{\Psi}\left(\hat{K}, X\right)\right|^2 \, d\hat{K}
$$

and the average invested capital per firm for sector X is:

$$
K_X = \frac{\int \hat{K} \left|\hat{\Psi}\left(\hat{K}, X\right)\right|^2 \, d\hat{K}}{N(X)} \tag{48}
$$

Note that this K_X is also equal to the average physical capital per firm for sector X, i.e. :

$$
K_X = \frac{\int K \, |\Psi(K, X)|^2 \, dK}{N(X)} \tag{49}
$$

Indeed, given our assumptions, the total physical capital is equal to the total capital invested:

$$
\int K \, |\Psi(K, X)|^2 \, dK = \int \hat{K} \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 \, d\hat{K}
$$

In the following, we will use alternately both expressions (48) or (49).

Ultimately, given a collective state, the distribution per sector of both invested capital and capital per firm are given by $\frac{\left|\hat{\Psi}\left(\hat{K}, X\right)\right|^2}{\hat{N}(X)}$ and $\frac{|\Psi(K, X)|^2}{N(X)}$ respectively.

Gathering equations (46), (47) and (48), we thus conclude that each collective state is singularly determined by the collection of data that characterizes each sector: the number of firms for each sector, the

number of investors for each sector, the average capital for each sector and the density of distribution of capital in each sector.

All the above quantities allow to study the capital allocation among sectors as well as its dependency in system parameters such as expected long-term return, short-term return, or any parameters involved in the model.

At the individual level At the individual level, the field formalism allows to compute agents' individual dynamics in the state defined by the background fields, through the transition functions of the system. These transition functions are themselves derived using the Green functions' formalism (see GLW). This study is left for a subsequent work. In the following, we solve the system for the background fields and compute the average associated quantities. This "static" point of view, will be extended by introducing some fluctuations in the expectations, leading to a dynamic of the average capital at the macro-level.

7   Resolution of the Model

Now that our initial framework has been turned into a proper field formalism, we can turn to the resolution of the model. More specifically, we want to find the background fields of the system. We start with some preliminary simplifications.

7.1   Preliminaries

To do so, we must find the configurations $\Psi(K, X)$ and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ of the fields that minimize S. To study the impact of financial on the real sector, we first minimize $S_1 + S_2$, then solve for the field $\Psi(K, X)$ as a function of the investors' variables. We then minimize $S_3 + S_4$, and find the minimal configuration of the investors' field $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$.

At this point, we can introduce a simplification and assume that investors invest in only one sector. This translates into the following condition:

$$G\left(X - \hat{X}\right) = \delta\left(X - \hat{X}\right) \tag{50}$$

This simplification does not reduce the generality of our model: actually, an investor acting in several sectors could be modelled as an aggregation of several investors. Nor does it mean that investors should be static, since they can still move from one sector to another.

The intermediate functions (43), (44) and (45) involved in the definition of the action functional (42) thus become:

$$u\left(K, X, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon}\left(K - \int \frac{F_2\left(R\left(K, X\right)\right)}{\int F_2\left(R\left(K', X\right)\right)\|\Psi\left(K', X\right)\|^2 dK'}\hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}\right) \tag{51}$$

$$f\left(\hat{X}, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon}\int\left(r\left(K, X\right) - \gamma\frac{\int K'\|\Psi\left(K', X\right)\|^2}{K} + F_1\left(\frac{R\left(K, X\right)}{\int R\left(K', X'\right)\|\Psi\left(K', X'\right)\|^2 d\left(K', X'\right)}, \Gamma\left(K, X\right)\right)\right)$$

$$\times \frac{F_2\left(R\left(K, \hat{X}\right)\right)}{\int F_2\left(R\left(K', \hat{X}\right)\right)\left\|\Psi\left(K', \hat{X}\right)\right\|^2 dK'}\left\|\Psi\left(K, \hat{X}\right)\right\|^2 dK$$

$$g\left(K, \hat{X}, \Psi, \hat{\Psi}\right) = \int\left(\nabla_{\hat{X}}F_0\left(R\left(K, \hat{X}\right)\right) + \nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K, \hat{X}\right)}{\int R\left(K', X'\right)\|\Psi\left(K', X'\right)\|^2 d\left(K', X'\right)}\right)\right)\frac{\left\|\Psi\left(K, \hat{X}\right)\right\|^2 dK}{\int\left\|\Psi\left(K', \hat{X}\right)\right\|^2 dK'}$$

where:

$$\int \frac{F_2\left(R\left(K, X\right)\right)G\left(X - \hat{X}\right)}{K\int F_2\left(R\left(K, X\right)\right)G\left(X - \hat{X}\right)\|\Psi\left(K, X\right)\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right) - 1$$

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7.2 Background field for the real economy

7.2.1 General formula

We first compute[6] the field of the real economy $\Psi(K, X)$ as a function of the field of the financial sector $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ by minimizing the (K, X) part of equation (42):

$$
\begin{aligned}
S_1 + S_2 = {} & -\int \Psi^{\dagger}(K, X) \left(\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K)\right) - \tau \left(\int |\Psi(K', X)|^2 \, dK'\right)\right. \\
& \left. + \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u\left(K, X, \Psi, \hat{\Psi}\right)\right)\right) \Psi(K, X) \, dK dX
\end{aligned}
\tag{52}
$$

For relatively slow fluctuations in X, the background fields $\Psi(K, X)$ and $\Psi^{\dagger}(K, X)$ decompose as a product:

$$
\begin{aligned}
\Psi(K, X) = {} & \exp\left(\int^X \frac{\nabla_X R(X)}{\sigma_X^2} H\left(\frac{\int \hat{K} \left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right) \\
& \times \exp\left(\int \left(K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))}\right) dK\right) \Psi(X) \Psi_1(K - K_X)
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\Psi^{\dagger}(K, X) = {} & \exp\left(-\int^X \frac{\nabla_X R(X)}{\sigma_X^2} H\left(\frac{\int \hat{K} \left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right) \\
& \times \exp\left(-\int \left(K - \frac{F_2(R(K, X)) K_X}{F_2(R(K_X, X))}\right) dK\right) \Psi^{\dagger}(X) \Psi_1^{\dagger}(K - K_X)
\end{aligned}
\tag{54}
$$

where K_X, the average invested capital per firm in sector X, is given by:

$$
K_X = \frac{\int \hat{K} \left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}
\tag{55}
$$

7.2.2 Determination of $\Psi(X)$ and $\|\Psi(X)\|^2$

The function $\Psi(X)$, that arises in the definitions (53) and (54) of the background fields $\Psi(K, X)$ and $\Psi^{\dagger}(K, X)$, minimizes:

$$
\begin{aligned}
& \int \Psi^{\dagger}(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{(\nabla_X R(X) H(K_X))^2}{2\sigma_X^2} + \frac{\nabla_X^2 R(K, X)}{2} H(K) + 2\tau |\Psi(X)|^2\right) \Psi(X) \\
& + D\left(\|\Psi\|^2\right) \left(\int \|\Psi(X)\|^2 - N\right) + \int \mu(X) \|\Psi(X)\|^2
\end{aligned}
\tag{56}
$$

where the constants $D\left(\|\Psi\|^2\right)$ and $\mu(X)$ are Lagrange multipliers, and implement the constraints:

$$
\int \|\Psi(X)\|^2 = N
$$

where N is the number of firms, and:

$$
\|\Psi(X)\|^2 \geqslant 0
$$

Incidentally, note that, to keep track of the dependency of the Lagrange multiplier in $\|\Psi\|^2$ in the above, we have chosen the notation $D\left(\|\Psi\|^2\right)$.

---

[6] For detailed computations of this subsection, see appendix 2.

The solution of the minimization problem for equation (56) is a function:

$$\Psi\left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}}\right)$$

It depends on the parameters of the system: the average invested capital, and the value of the expected return and its derivatives. The parameter $\hat{K}_{X,0}$ is a normalization factor. Solving equation (56) in the limit of small fluctuations σ_X^2 in X , the minimization of equation (56) becomes:

$$D\left(\|\Psi\|^2\right) = 2\tau \|\Psi(X)\|^2 + \frac{1}{2}\left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)}\right) H^2\left(\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)\hat{K}_X}{H\left(\hat{K}_X\right)\|\Psi(X)\|^2}\right)$$
(57a)

where the Lagrange multiplier $D\left(\|\Psi\|^2\right)$ satisfies:

$$ND\left(\|\Psi\|^2\right) = 2\tau \int |\Psi(X)|^4 + \frac{1}{2}\int\left(\nabla_X R(X) H\left(\frac{\int \hat{K} \left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \|\Psi(X)\|^2$$
(58)

The two equations above, (57a) and (58), can be solved as a function of the total capital invested in sector X, $\hat{K}_X$ for the particular form of the function H. Two examples of such resolution are given in appendix 2.2.

However a precise solution of (58) is useless in the following, since we will rather need the first equation, (57a), to be written as:

$$\|\Psi(X)\|^2 = (2\tau)^{-1}\left(D\left(\|\Psi\|^2\right) - \frac{1}{2}\left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)}\right) H^2(K_X)\left(1 - \frac{H'\left(\hat{K}_X\right)K_X}{H\left(\hat{K}_X\right)}\right)\right)$$
(59)

This last formula will be used extensively in the sequel to compute K_X the average physical capital per firm in sector X.

7.2.3   Determination of $\Psi_1(K - K_X)$

The function $\Psi_1(K - K_X)$, involved in the definitions (53) and (54) of the background fields $\Psi(K, X)$ and $\Psi^\dagger(K, X)$, describes the fluctuations of capital around an average value K_X. It satisfies the equation:

$$-\nabla_K^2 \Psi_1(K - K_X) + \left(K - \frac{F_2(R(K, X))K_X}{F_2(R(K_X, X))}\right)^2 \Psi_1(K - K_X) = 0$$
(60)

with solution:

$$\Psi_1(K - K_X) = \mathcal{N} \exp\left(-\left(K - \frac{F_2(R(K, X))K_X}{F_2(R(K_X, X))}\right)^2\right)$$
(61)

where $\mathcal{N}$ is a normalization factor. The same formula holds for $\Psi_1^\dagger(K - K_X)$.

7.2.4   Interpretation of the formula and density of agents

Formulas (53) and (54) decompose in four terms:

The two exponential factors encapsulate the agents' average move towards sector X and capital accumulation at this point, respectively.

The two factors $\Psi(X)\Psi_1(K - K_X)$ and $\Psi^\dagger(X)\Psi_1^\dagger(K - K_X)$ translate the fact that in the sector's space, motion is slower than capital accumulation: capital accumulates as a function of the position X of the sector, through the capital allocated in this sector, K_X. The repartition of agents in space is described by the squared field $|\Psi(X)|^2$ (see (46)).

Formula (61) shows that the capital accumulated by a firm in a sector X is centered around the average capital K_X in this sector, weighted by a factor $\frac{F_2(R(K,X))}{F_2(R(K_X,X))}$ that depends on the firm's expected long-term return, and is relative to the average expected long-term return of the whole sector X. The latter is described by the function $F_2\left(R\left(K_X,X\right)\right)$[7].

Once the solution of (56) found, it can be used alongside equation (61) to compute $\|\Psi\left(K,X\right)\|^2$ in the limit of relatively small fluctuations σ_X^2 in X. We obtain:

$$\|\Psi\left(K,X\right)\|^2 = \mathcal{N}\left\|\Psi\left(X,\left(\nabla_X R\left(X\right)\right)^2, \frac{\hat{K}_X}{\hat{K}_{X,0}}\right)\right\|^2 \times \exp\left(-\left(K-K_X\right)^2\right) \tag{62}$$

Note that the form of the exponential in (62) implies that the average physical capital is given by K_X, i.e. the capital invested in sector X, divided by the number of firms in this sector, which computes the density of agents defined by a given position X and capital K. Mathematically:

$$\int K\,\|\Psi\left(K,X\right)\|^2\,d\hat{K} = \int \hat{K}\,\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2\,d\hat{K}$$

7.3 Background field for the financial sector

In this paragraph we compute the background field $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ representing the financial markets.

7.3.1 Minimization of $S_3 + S_4$

Once the background fields for the real economy $\Psi\left(K,X\right)$ and $\Psi^\dagger\left(X,K\right)$ found, the minimization problem for $S_3 + S_4$ can be considered. The computations are presented in appendix 3.

Given the form of $\Psi\left(X,K\right)$ and $\Psi^\dagger\left(X,K\right)$ (53) and (54), the field action for the background field of the financial markets $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ can be reduced to the expression (see appendix 3.1.1):

$$S_3 + S_4 = -\int \hat{\Psi}^\dagger\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \tag{63}$$

with:

$$f\left(\hat{X},K_{\hat{X}}\right) = \frac{1}{\varepsilon}\left(r\left(K_{\hat{X}},\hat{X}\right) - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\|\Psi\left(X'\right)\|^2\,dX'}\right)\right) \tag{64}$$

$$g\left(\hat{X},K_{\hat{X}}\right) = \left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right\|} + \nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\|\Psi\left(X'\right)\|^2\,dX'}\right)\right) \tag{65}$$

The first order condition for (63) is computed using a change of variable (see appendix 3.1.2):

$$\hat{\Psi} \to \exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int g\left(\hat{X}\right)d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\hat{\Psi}$$

which yields the following equations for $\hat{\Psi}$ and $\hat{\Psi}^\dagger$:

$$0 = \left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2 - \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} \tag{66}$$

$$+ \left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right) - \hat{\lambda}\right)\hat{\Psi} - F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi}$$

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[7]See discussion below equation (5).

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$$0 \quad = \quad \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g \left(\hat{X}, K_{\hat{X}} \right) \right)^2 - \frac{1}{2} \nabla_{\hat{X}} g \left(\hat{X}, K_{\hat{X}} \right) \right) \hat{\Psi}^{\dagger} \tag{67}$$

$$+ \left(\left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} + \hat{K} f \left(\hat{X}, K_{\hat{X}} \right) \right) \nabla_{\hat{K}} - \hat{\lambda} \right) \hat{\Psi} - F \left(\hat{X}, K_{\hat{X}} \right) \hat{K} \hat{\Psi}^{\dagger}$$

with:

$$F \left(\hat{X}, K_{\hat{X}} \right) \quad = \quad \nabla_{K_{\hat{X}}} \left(\frac{\left(g \left(\hat{X}, K_{\hat{X}} \right) \right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g \left(\hat{X}, K_{\hat{X}} \right) + f \left(\hat{X}, K_{\hat{X}} \right) \right) \frac{\left\| \hat{\Psi} \left(\hat{X} \right) \right\|^2}{\left\| \Psi \left(\hat{X} \right) \right\|^2} \tag{68}$$

$$+ \frac{\nabla_{K_{\hat{X}}} f^2 \left(\hat{X}, K_{\hat{X}} \right)}{\sigma_{\hat{K}}^2 \left\| \Psi \left(\hat{X} \right) \right\|^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}}$$

where $\left\langle \hat{K}^2 \right\rangle_{\hat{X}}$ denotes the average of $\hat{K}^2$ in sector $\hat{X}$ (see appendix 3.1.2) and $\left\| \hat{\Psi} \left(\hat{X} \right) \right\|^2 = \int \left\| \hat{\Psi} \left(\hat{X}, \hat{K} \right) \right\|^2 d\hat{K}$. To close the system of equations (66) and (250) for $\hat{\Psi}$ and $\hat{\Psi}^{\dagger}$, we add the constraint implemented by the Lagrange multiplier $\hat{\lambda}$:

$$\int \left\| \hat{\Psi} \left(\hat{X}, \hat{K} \right) \right\|^2 d\hat{X} d\hat{K} = \hat{N} \tag{69}$$

Note that the function $F \left(\hat{X}, K_{\hat{X}} \right)$ in (68), which arises in the minimization equations (66) and (250), describes the impact of individual variations on the collective state or field $\hat{\Psi}$, and can be neglected in first approximation.

7.3.2 Formula for the background fields

The solutions of equations (66) and (250) are computed in the limit of small fluctuations in $\hat{X}$ (see appendix 3.1.3). We find an infinite number of solutions parametrized by $\hat{\lambda}$:

$$\hat{\Psi}_{\hat{\lambda}} \left(\hat{X}, \hat{K} \right) \quad = \quad \hat{\Psi}_{\hat{\lambda}}^{(1)} \left(\hat{X}, \hat{K} \right) \exp \left(\frac{1}{\sigma_{\hat{X}}^2} \int g \left(\hat{X} \right) d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f \left(\hat{X} \right) \right) \tag{70}$$

$$\times D_{p(\hat{X}, \hat{\lambda})} \left(\frac{\left(\left| f \left(\hat{X} \right) \right| \right)^{\frac{1}{2}}}{\sigma_{\hat{K}}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}, K_{\hat{X}} \right)}{f^2 \left(\hat{X} \right)} \right) \right)$$

and:

$$\hat{\Psi}_{\hat{\lambda}}^{\dagger} \left(\hat{X}, \hat{K} \right) = \hat{\Psi}_{\hat{\lambda}}^{(1)\dagger} \left(\hat{X}, \hat{K} \right) \exp \left(- \left(\frac{1}{\sigma_{\hat{X}}^2} \int g \left(\hat{X} \right) d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f \left(\hat{X} \right) \right) \right) D_{p(\hat{X}, \hat{\lambda})} \left(\frac{\hat{K}}{\sigma_{\hat{K}}} \left(\left| f \left(\hat{X} \right) \right| \right)^{\frac{1}{2}} \right) \tag{71}$$

with:

$$p \left(\hat{X}, \hat{\lambda} \right) = - \frac{\left(g \left(\hat{X} \right) \right)^2 + \sigma_{\hat{X}}^2 \left(f \left(\hat{X} \right) + \nabla_{\hat{X}} g \left(\hat{X}, K_{\hat{X}} \right) - \frac{\sigma_{\hat{K}}^2 F^2 \left(\hat{X}, K_{\hat{X}} \right)}{2 f^2 \left(\hat{X} \right)} + \hat{\lambda} \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2 \left(\hat{X} \right)}} - \frac{1}{2} \tag{72}$$

and where D_p is the parabolic cylinder function with parameter p. The fields $\hat{\Psi}_{\hat{\lambda}}^{(1)} \left(\hat{X}, \hat{K} \right)$ and $\hat{\Psi}_{\hat{\lambda}}^{(1)\dagger} \left(\hat{X}, \hat{K} \right)$ are corrections in σ_X^2:

$$\hat{\Psi}_{\hat{\lambda}}^{(1)} \left(\hat{X} \right) = \sqrt{C \left(\hat{\lambda} \right)} \exp \left(- \int \frac{\left(\left(\hat{K}^2 \mp \frac{1}{4} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}, K_{\hat{X}} \right)}{f^2 \left(\hat{X} \right)} \right)^2 \right) f' \left(\hat{X} \right) \right)^2}{\sigma_{\hat{K}}^2 \left(2 \hat{K} f \left(\hat{X} \right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}, K_{\hat{X}} \right)}{f^2 \left(\hat{X} \right)} \right) \left| f \left(\hat{X} \right) \right| \right)} d\hat{K} \right) \tag{73}$$

and:

$$\hat{\Psi}^{(1)\dagger}\left(\hat{X}\right) = \sqrt{C\left(\hat{\lambda}\right)} \exp\left(\int \frac{\left(\left(\hat{K}^2 \pm \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'\left(\hat{X}\right)\right)^2}{\sigma_{\hat{K}}^2\left(2\hat{K}f\left(\hat{X}\right) + \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)\left|f\left(\hat{X}\right)\right|\right)} d\hat{K}\right) \tag{74}$$

where the symbol $\pm$ stands for $sign\left(f\left(\hat{X}\right)\right)$, $\mp$ for $-sign\left(f\left(\hat{X}\right)\right)$, and where $C\left(\hat{\lambda}\right)$ is a function of $\hat{\lambda}$ obtained by imposing the normalization condition (69) for $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)$, i.e.:

$$\int \left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right) = \hat{N}$$

Appendix 3.1.4.2 shows that:

$$C\left(\hat{\lambda}\right) \equiv C\left(\bar{p}\left(\hat{\lambda}\right)\right) \simeq \frac{\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2\left(\frac{\left(\bar{p}(\hat{\lambda}) + \frac{1}{2}\right) f'(X_0)}{f(\hat{X}_0)}\right)^2}{96|f(\hat{X}_0)|}\right) \hat{N}\Gamma\left(-\bar{p}\left(\hat{\lambda}\right)\right)}{\left(\frac{\langle|f(\hat{X})|\rangle}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} V_r\left(\text{Psi}\left(-\frac{\bar{p}(\hat{\lambda})-1}{2}\right) - \text{Psi}\left(-\frac{\bar{p}(\hat{\lambda})}{2}\right)\right)} \tag{75}$$

where:

$$\bar{p}\left(\hat{\lambda}\right) = \left(-\frac{\frac{(g(\hat{X}_0))^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}_0\right) + \frac{1}{2}\left|f\left(\hat{X}_0\right)\right| + \nabla_{\hat{X}} g\left(\hat{X}_0, K_{\hat{X}_0}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}_0, K_{\hat{X}_0}\right)}{2f^2(\hat{X}_0)} + \hat{\lambda}\right)}{\left|f\left(\hat{X}_0\right)\right|}\right) \tag{76}$$

and:

$$\hat{X}_0 = \arg\min_{\hat{X}} \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2\left(\frac{\left(p(\hat{\lambda}) + \frac{1}{2}\right) f'(X)}{f(\hat{X})}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\right) \tag{77}$$

The solutions (70) and (71) are thus a one-parameter $\hat{\lambda}$-family of solutions to (66). In other words, there is an infinite number of background fields, but in the following, only one will eventually be relevant.

7.3.3 Density of investors

Finally, we find the density of investors at a given position $\hat{X}$ and capital $\hat{K}$ in the background field or collective state $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)$:

$$\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 \simeq C\left(\bar{p}\left(\hat{\lambda}\right)\right) \exp\left(-\frac{\sigma_X^2 \hat{K}^4 \left(f'(X)\right)^2}{96\sigma_{\hat{K}}^2\left|f\left(\hat{X}\right)\right|}\right) \tag{78}$$

$$\times D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)$$

Depending on the value of $\hat{\lambda}$, there is a multiplicity of background fields, each contributing with its own weight $\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2$, defined in (78), to the computation of the average capital invested.

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7.3.4 Estimation of $S_3 + S_4$ for the background field

We conclude this paragraph by computing an estimation of $S_3 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right)$ for any background field $\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right)$ defined in (70). This estimation will be later used to discriminate between the potential configurations $\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right)$ for the background field.

In order to do so, we multiply (66) by $\hat{\Psi}_{\hat{\lambda}}^{\dagger} \left(\hat{K}, \hat{X} \right)$ on the left and integrate the equation over $\hat{K}$ and $\hat{X}$. It yields:

$$0 = S_3 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) - \hat{\lambda} \int \left\| \hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right\|^2 d\hat{K} d\hat{X} - \int F \left(\hat{X}, K_{\hat{X}} \right) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right\|^2 d\hat{K} d\hat{X}$$

Using the constraint imposed on the number of investors:

$$\int \left\| \hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right\|^2 d\hat{K} = \hat{N}$$

we find:

$$S_3 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) = \hat{\lambda} \hat{N} + \int F \left(\hat{X}, K_{\hat{X}} \right) \hat{K} \left\| \hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right\|^2 d\hat{K} d\hat{X} \tag{79}$$

Then, estimating the second term of the rhs of (79), we show in appendix 3.1.4 that ultimately:

$$S_3 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right) \right) \simeq - \left(\left| \hat{\lambda} \right| - M \right) \hat{N} \tag{80}$$

where M, the lowest bound for the eigenvalues $\left| \hat{\lambda} \right|$ will be defined in the next paragraph.

7.4 Average capital invested

7.4.1 General equation

Now that we have computed both the background fields for the real and the financial sector of the economy, we can determine $K_{\hat{X}}$, the average capital per firm in sector $\hat{X}$, in this environment.

To do so, we first rewrite the defining equation of $K_{\hat{X}}$, given in (55), as the following identity:

$$K_{\hat{X}} \left\| \Psi \left(\hat{X} \right) \right\|^2 = \int \hat{K} \left\| \hat{\Psi} \left(\hat{K}, \hat{X} \right) \right\|^2 d\hat{K} \tag{81}$$

This equation is actually an equation for $K_{\hat{X}}$. Actually, we have expressed in equation (59) the squared background field $\left\| \Psi \left(\hat{X} \right) \right\|^2$ as a function of $K_{\hat{X}}$, and the field $\hat{\Psi} \left(\hat{K}, \hat{X} \right)$, and $\hat{\Psi} \left(\hat{K}, \hat{X} \right)$ is itself a function of $K_{\hat{X}}$ through equation (78).

7.4.2 Averaging over background fields

Note that the equation (81) above is defined for a given background field $\hat{\Psi} \left(\hat{K}, \hat{X} \right)$, whereas equation (78) in the previous paragraph yields an infinity of background fields $\hat{\Psi}_{\hat{\lambda}} \left(\hat{K}, \hat{X} \right)$ indexed by the variable $\hat{\lambda}$. To account for this infinite number of solutions, we must average the rhs of (81) over $\hat{\lambda}$. The weight associated to a particular value of $\hat{\lambda}$, which computes the probability of the configuration $\hat{\Psi}_{\hat{\lambda}}$, is given by:

$$\exp \left(- \left(S_3 \left(\hat{\Psi}_{\hat{\lambda}} \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \right) \right) \right) \tag{82}$$

Given the minimization equation (66) that defines $\hat{\Psi}_{\hat{\lambda}}$, we have seen that (see (80)):

$$S_3 \left(\hat{\Psi}_{\hat{\lambda}} \right) + S_4 \left(\hat{\Psi}_{\hat{\lambda}} \right) \simeq \left(\left| \hat{\lambda} \right| - M \right) \hat{N}$$

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where the last equality implements the constraint that the number of investors should be $\hat{N}$. As a consequence, the weight (82) reduces to:

$$\exp\left(\hat{\lambda}\int\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^{2}\right)=\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)$$

and equation (81) is replaced, up to a normalization factor, by:

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\int\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},X\right)\right\|^{2}d\left|\hat{\lambda}\right|d\hat{K} \tag{83}$$

7.4.3   Computing the integral over $\left|\hat{\lambda}\right|$

Appendix 3.1.4.2 shows that the integral over $\left|\hat{\lambda}\right|$ has a lower bound M. This lower bound[8] is defined by:

$$M=\max_{\hat{X}}A\left(\hat{X}\right) \tag{84}$$

with[9]:

$$A\left(\hat{X}\right)=\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^{2}\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)} \tag{85}$$

Incidentally, an interpretation of $A\left(\hat{X}\right)$ and its bound M will be given in section 8.2.1.

Once the parameter M found, the integral in (83) becomes:

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\int_{\left|\hat{\lambda}\right|>M}\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},X\right)\right\|^{2}d\left|\hat{\lambda}\right|d\hat{K} \tag{86}$$

However, since the number of investors $\hat{N}$ is very large, $\hat{N}>>1$, the integral in (86) is peaked at the minimum value for $\left|\hat{\lambda}\right|$, that is M, and K_{X} satisfies:

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\left\|\hat{\Psi}_{p(-M)}\left(\hat{K},\hat{X}\right)\right\|^{2}d\hat{K} \tag{87}$$

If we define $\bar{p}=\bar{p}\left(-M\right)$, $p=p\left(-M\right)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)=\hat{\Psi}_{p(-M)}\left(\hat{K},\hat{X}\right)$, we can rewrite equation (87) as:

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\left\|\hat{\Psi}_{p}\left(\hat{K},\hat{X}\right)\right\|^{2}d\hat{K} \tag{88}$$

7.4.4   Final form of the capital equation

Using the form of the financial background field (78) to compute the integral in (88), the equation defining average capital at point $\hat{X}$ (see appendix 3.1.4.2) ultimately becomes:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{89}$$

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[8]This lower bound is reminiscent of the fact that the Lagrange multiplier λ is the eigenvalue of the second order operator arising in equation (67), and that this operator is bounded from below.

[9]In this formula, the term $\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}$ can be neglected in general (see appendix 2). In the sequel, this term will often be omitted.

with:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right) = \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 (f'(X))^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \tag{90}$$
$$\times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma(-p-1)\Gamma(-p)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma(-p)\Gamma(-p+1)}\right)$$

and:

$$p = \frac{M - \left(\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}} \tag{91}$$

It is this equation (89) that will be central to our following computations. We will give in section 8.2.1 an interpretation of p in terms of relative attractivity of a firm vis a vis its neighbours.

To conclude, note that for later purposes that the function $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ is asymptotically given by:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right) \sim_\infty \alpha\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 (f'(X))^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right) \tag{92}$$

with:

$$\alpha \simeq \frac{3}{2}$$

Equation (89) involves functions that have a general form, such as f, and complicated functions of the unknown variable $K_{\hat{X}}$ (see equation (72)). As such, it cannot usually be solved analytically. We will now provide several method to approach the solutions (89).

8 Finding the average capital in a given environment

Except for particular cases[10], equation (89) cannot be solved analytically. However, several complementary approaches can be used to shed light on the behaviour of its solutions. The first approach is the most general: we study the differential form of (89) out of any assumptions on the parameter-functions. This allows to compute, for each sector $\hat{X}$, the derivative of the average capital per firm $K_{\hat{X}}$ with respect to any parameter. In particular, the derivative of the sector expected returns relative to its neighbours can be computed, which shows the influence of the local environment on a sector. However, this first approach does not yield the precise level of capital for each sector. A second approach considers the expansion of (89) around particular values of capital, and a third approximates the resolution of equation (89) for standard forms of the parameter-functions. Combined, they confirm and precise our initial results.

We will check in the two last approaches that the equation (89) for average capital has several solutions, for a given set of parameters functions, and that as such, background fields $\hat{\Psi}_p\left(\hat{K}, \hat{X}\right)$ are not unique. An infinite number of collective state may arise, depending on some initial configuration. We will discuss this point in section 10.

8.1 First approach: differential form

One way to better understand equation (89) is to study its differential form.

Assume at point $\hat{X}$ of the system, a variation $\delta Y\left(\hat{X}\right)$ for any parameter, in which the parameter $Y\left(\hat{X}\right)$ can be either $R(X)$, its gradient, or any parameter arising in the definition of f and g. This variation

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[10] These particular cases will be studied in the following sections.

$\delta Y\left(\hat{X}\right)$ itself induces a variation $\delta K_{\hat{X}}$ in the average capital, variation whose expression is obtained by differentiating equation (89):

$$\delta K_{\hat{X}} = \left(-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + l\left(\hat{X},K_{\hat{X}}\right)\right) + k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}}$$

$$+\frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)\delta Y\left(\hat{X}\right) \tag{93}$$

8.1.1 Developed form of the differential form (93)

For the later use of equation (93) we give here its developed form (see appendix 3.2.1):

$$\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} = \left(k\left(p\right)\frac{\partial_Y\left(M - \left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)}\right.$$

$$\left. -\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial Y}\left(1 + \left(p + \mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right) + \frac{1}{2}\right)k\left(p\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial Y}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + m_Y\left(\hat{X},K_{\hat{X}}\right)\right)\right)\frac{\delta Y}{D} \tag{94}$$

with $\mathcal{H}$ the Heaviside function, and where we have defined the intermediate functions:

$$D = 1 + \left(\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}\left(1 + \left(p + \mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right) + \frac{1}{2}\right)k\left(p\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + l\left(\hat{X},K_{\hat{X}}\right)\right)\right.$$

$$\left. -k\left(p\right)\frac{\partial_{K_{\hat{X}}}\left(M - \left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)}\right)K_{\hat{X}} \tag{95}$$

$$k\left(p\right) = \frac{\frac{d}{dp}\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\hat{\Gamma}\left(p+\frac{1}{2}\right)} \sim_\infty \sqrt{\frac{p-\frac{1}{2}}{2}} - \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3} \tag{96}$$

$$l\left(\hat{X},K_{\hat{X}}\right) = \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}$$

$$+\frac{\partial p}{\partial K_{\hat{X}}}\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}$$

and:

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$$m_Y\left(\hat{X}, K_{\hat{X}}\right) \;=\; \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(\nabla_Y \left(f'\left(\hat{X}\right)\right)^2 \left|f\left(\hat{X}\right)\right| - 3\left(\nabla_Y \left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}$$

$$+\nabla_Y p\,\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}$$

The interpretation of these intermediate functions will be given later in section 10.2.1.

Recall that equation (94) is the differentiated version of the equation defining the average capital $K_{\hat{X}}$, equation (89). It will be used to compute the dependency of the average per firm in sector $\hat{X}$, $K_{\hat{X}}$, as a function of any parameter $Y(\hat{X})$. But more fundamentally, it allows to understand the solutions of equation (89) and their stability with respect to the parameters' variations.

8.1.2   Local stability

The differential form given by equation (93), computes the effect of a variation $\delta Y\left(\hat{X}\right)$ in the parameters on the average capital $K_{\hat{X}}$. Actually, equation (93) can be understood as the fixed-point equation of a dynamical system through the following mechanism.

Each variation $\delta Y\left(\hat{X}\right)$ in the parameters impacts the average capital, which must then be computed with the new parameters. The first change induced is written $\delta K_{\hat{X}}^{(1)}$:

$$\delta K_{\hat{X}}^{(1)} = \frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X}, K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right)\delta Y\left(\hat{X}\right) \tag{97}$$

In a second step, the variation $\delta K_{\hat{X}}$ impacts the various functions implied in (89), and indirectly modifies $K_{\hat{X}}$ through the first term in the rhs of (93):

$$\left(-\left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right) + k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}}^{(1)} \tag{98}$$

These two effects combined, (97) and (98), yield the total variation $\delta K_{\hat{X}}$.

Importantly, note that if we can interpret $\delta K_{\hat{X}}^{(1)}$ as a variation at time t, we can also infer from the indirect effect (98) that $\delta K_{\hat{X}}$ is itself a variation at time $t+1$. Equation (93) can thus be seen as the fixed point equation of a dynamical system written:

$$\delta K_{\hat{X}}\left(t+1\right) = \left(-\left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right) + l\left(\hat{X}, K_{\hat{X}}\right)\right) + k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}}\left(t\right) \tag{99}$$

$$+\frac{\partial}{\partial Y\left(\hat{X}, t\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X}, K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right)\delta Y\left(\hat{X}, t\right)$$

whose fixed point is the solution of (93):

$$\delta K_{\hat{X}} = \frac{\frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X}, K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right)}{1+\left(\left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right) + l\left(\hat{X}, K_{\hat{X}}\right)\right) - k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}}\delta Y\left(\hat{X}\right) \tag{100}$$

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This solution (100) is stable when:

$$\left| k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}} - \left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial \left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + l\left(\hat{X},K_{\hat{X}}\right)\right)\right| < 1 \tag{101a}$$

i.e. when D, defined in (95), is positive, and unstable otherwise. So that the stability of this average capital depends, in last analysis, on the sign of D.

The notion of stability reveals the two types of solution in the model, the solutions for the average capital per firm $K_{\hat{X}}$ solution of (89): the stable solutions $K_{\hat{X}}$ can be considered as sector $\hat{X}$'s equilibrium averages. However unstable solutions must rather be considered as thresholds: when $K_{\hat{X}}$ is driven away from this threshold, it may either converge toward a stable solution of (89), or diverge towards 0 or infinity.

Introducing the dynamical system (99) may at first sight seem artificial, but it is nonetheless coherent within the context of our field model. Actually, this arbitrary variation $\delta K_{\hat{X}}^{(1)}$ induced by a change in parameter actually reveals a shift $\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)$ in the background state $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. Since there is no reason for the new configuration $\hat{\Psi}\left(\hat{K},\hat{X}\right) + \delta\hat{\Psi}\left(\hat{K},\hat{X}\right)$ to be a minimum of the action functional, we must determine whether the system will settle on a slightly modified background state with a different $K_{\hat{X}}$, or be driven towards a different equilibrium. To do so, we must study the dynamics equation for $K_{\hat{X}}$ (99).

8.1.3   Applications of the differential form

Once the notion of stability understood, we can use equation (94) to compute the impact of the variation of any parameter $Y(\hat{X})$ on $\delta K_{\hat{X}}$. Two applications are of particular interest to us.

Main application   The main application of equation (94) is to consider a parameter denoted $Y(\hat{X})$, that encompasses the relative expected returns of sector X vis-à-vis its neighbouring sectors, and defined as:

$$Y(\hat{X}) = M - \left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right) \tag{102}$$

This parameter enters directly in (94), the differentiated equation for $K_{\hat{X}}$, through parameter (72). It is composed of three terms.

The first term, $\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}$, is directly proportional to the gradient of expected long-term returns $\nabla R\left(K_{\hat{X}},\hat{X}\right)$[11]. It is minimal for an extremum of the expected return $R\left(K_{\hat{X}},\hat{X}\right)$.

The second term, $\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)$, is proportional to the second derivative $\nabla^2 R\left(K_{\hat{X}},\hat{X}\right)$ of $R\left(K_{\hat{X}},\hat{X}\right)$, and is minimal when expected returns are maximum.

The third term, $-\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$, is a corrective term. It can be neglected in first approximation[12], but will be interpreted in section.

As stated previously, the parameter $Y(\hat{X})$ is thus a measure of the expected long-term return of a sector relative to its neighbours: it is a local maximum when $R\left(K_{\hat{X}},\hat{X}\right)$ is itself a local maximum.

Using (94), we have:

$$\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} = -\frac{\frac{k(p)}{f\left(\hat{X},K_{\hat{X}}\right)}K_{\hat{X}}}{D}\delta Y\left(\hat{X}\right) \tag{103}$$

---

[11] See the definition of the parameter function g, equation (65).
[12] See the discussion following equation (68).

Given equation (96), $k(p)$ is positive at the first order in σ_X^2. More precisely, using equation (96):

$$k(p) \sim_\infty \sqrt{\frac{p - \frac{1}{2}}{2} - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) (f'(X))^2}{48 \left| f\left(\hat{X}\right) \right|^3}}$$

along with equation (72), we can infer that $\sqrt{\frac{p-\frac{1}{2}}{2}}$ is of order $\frac{1}{\sigma_X}$ and $\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)\left(f'(X)\right)^2}{48|f(\hat{X})|^3} \sim 1$.

So that in a stable equilibrium, i.e. for $D > 0$, equation (103) implies that the dependency of $K_{\hat{X}}$ in the parameter $A\left(\hat{X}\right)$ is negative:

$$\frac{\delta K_{\hat{X}}}{\delta A\left(\hat{X}\right)} < 0$$

We have seen above that $Y\left(\hat{X}\right)$ is minimal for a maximum expected long-term return $R\left(\hat{X}, K_{\hat{X}}\right)$: when the equilibrium is stable, capital accumulation is maximal for sectors that are themselves a local maximum for $R\left(\hat{X}, K_{\hat{X}}\right)$.

On the other hand, when the equilibrium is unstable, i.e. for $D < 0$, the capital $K_{\hat{X}}$ is minimal for $R\left(\hat{X}, K_{\hat{X}}\right)$ maximal.

Actually, as seen above, in the instability range $D < 0$, the average capital $K_{\hat{X}}$ acts as a threshold. When, due to variations in the system's parameters, the average capital per firm is shifted above the threshold $K_{\hat{X}}$, capital will either move to the next stable equilibrium, possibly zero, or tend to infinity. Our results show that when the expected long-term return of a sector increases, the threshold $K_{\hat{X}}$ decreases, which favours capital accumulation.

Additional application A second use of equation (94) is to consider $Y(\hat{X})$ as any parameter-function involved in the definition of $f\left(\hat{X}, K_{\hat{X}}\right)$ that may condition either real short-term returns or the price-dividend ratio.

We can see that in this case, $Y(\hat{X})$ only impacts $f\left(\hat{X}, K_{\hat{X}}\right)$, so that equation (94) simplifies and yields:

$$
\begin{aligned}
\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} = & -\frac{m_Y\left(\hat{X}, K_{\hat{X}}\right)}{D} \delta Y \\
& -\frac{1}{D}\left(\frac{\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y}\left(1 + \left(p + H\left(f\left(\hat{X}, K_{\hat{X}}\right)\right) + \frac{1}{2}\right) k(p)\right)}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \|\Psi(\hat{X}, K_{\hat{X}})\|^2}{\partial Y}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}\right) \delta Y
\end{aligned}
\tag{104}
$$

Incidentally, note that p being proportional to $f^{-1}\left(\hat{X}\right)$, $m_Y\left(\hat{X}, K_{\hat{X}}\right)$ rewrites:

$$
\begin{aligned}
-m_Y\left(\hat{X}, K_{\hat{X}}\right) = & \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(3\left(\nabla_Y \left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2 - \nabla_Y\left(f'\left(\hat{X}\right)\right)^2 \left|f\left(\hat{X}\right)\right|\right)\left(p + \frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4} \\
& + \nabla_Y\left|f\left(\hat{X}\right)\right| \frac{\sigma_X^2 \sigma_{\hat{K}}^2 p\left(p + \frac{1}{2}\right)(f'(X))^2}{48\left|f\left(\hat{X}\right)\right|^4}
\end{aligned}
\tag{105}
$$

The first term in the rhs of (104) is the impact of an increase in investors' short-term returns. The second is the variation in capital needed to maintain investors' overall returns.

The sign of $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ given by equation (104) can be studied under two cases: the stable and the unstable equilibrium.

Let us first consider the case of a stable equilibrium, i.e. $D > 0$.

The first term in the rhs of (104), the variation induced by an increase in short-term returns, is in general positive for $f'\left(\hat{X}\right)$ proportional to $f\left(\hat{X}\right)$, that is for instance when the function $f\left(\hat{X}\right)$, that describes short-term returns and prices, depends on the variable $K_{\hat{X}}$ raised to some arbitrary power.

Indeed in that case:

$$3\left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2 - \nabla_Y\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| = \left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2$$

The second term in the rhs of (104) is in general negative. When $\frac{\partial f(\hat{X}, K_{\hat{X}})}{\partial Y} > 0$, i.e. when returns are increasing in Y, a rise in Y increases returns and decreases the capital needed to maintain these returns. Similarly, when $\frac{\partial \left\|\Psi(\hat{X}, K_{\hat{X}})\right\|^2}{\partial Y} > 0$, i.e. when the number of agents in sector $\hat{X}$ is increasing in Y, a rise in Y increases the number of agents that move towards point $\hat{X}$, and the average capital per firm diminishes.

The net variation (104) of $K_{\hat{X}}$ is the sum of these two contributions. Considering an expansion of (104) in powers of σ_X^2, the first contribution $-m_Y\left(\hat{X}, K_{\hat{X}}\right)$ is of magnitude $\left(\sigma_X^2\right)^{-1}$, whereas the second is proportional to $k(p) \sim (\sigma_X)^{-1}$. The variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is thus positive: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} > 0$. In most cases, a higher short-term return, decomposed as a sum of dividend and price variation, induces a higher average capital. This effect is magnified for larger levels of capital: the third approach will confirm that, in most cases, the return $f\left(\hat{X}\right)$ is asymptotically a constant $c << 1$ when capital is high: $K_{\hat{X}} >> 1$.

Turning now to the case of an unstable equilibrium, i.e. $D < 0$, the variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is negative: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} < 0$. In the instability range, and due to this very instability, an increase in returns $f\left(\hat{X}\right)$ reduces the threshold of capital accumulation for low levels of capital. When short-term returns $f\left(\hat{X}\right)$ increase, a lower average capital will trigger capital accumulation towards an equilibrium. Otherwise, when average capital $K_{\hat{X}}$ is below this threshold, it will converge toward 0.

8.2 Second approach: expansion around particular solutions

The second approach to equation (89) is to find the average capital at some particular points, and then by first order expansion, the solutions in the neighbourhood of these particular points. We choose as particular points the values $\hat{X}$ and $K_{\hat{X}}$ that maximize $A\left(\hat{X}\right)$, with:

$$A\left(\hat{X}\right) = \frac{\left(g\left(\hat{X}, K_{\hat{X}_M}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}, K_{\hat{X}_M}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}, K_{\hat{X}_M}\right)} + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}_M}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}_M}\right)}{2f^2\left(\hat{X}, K_{\hat{X}_M}\right)} \quad (106)$$

These points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$ are such that[13]:

$$A\left(\hat{X}_M\right) = M = \max_{\hat{X}} A\left(\hat{X}\right) \quad (107)$$

8.2.1 Interpretation of the particular points

We can interpret these particular points $\hat{X}_M$ by considering the function p. We have already defined p in (140), as:

$$p = \frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)}$$

Note that p is minimal and equal to 0 at points $\hat{X}_M$. The function p measures sector X long-term attractivity relative to its neighbouring sectors, normalized by its short run returns.

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[13] See equation (84).

This can be showed by discarding the corrective term $\frac{\sigma_K^2 F^2\left(\hat{X}, K_{\hat{X}_M}\right)}{2f^2\left(\hat{X}, K_{\hat{X}_M}\right)} << 1$ in (106), and we have in first approximation[14], up to a constant:

$$p = \frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)} \simeq \frac{M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}_M}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}_M}\right)\right)}{f\left(\hat{X}\right)} + \frac{3}{2} \tag{108}$$

Recall that $g\left(\hat{X}\right)$ is the investors' capital mobility at sector $\hat{X}$. When capital mobility is high, i.e. p is low, sector $\hat{X}$ is shunned by investors, who favour more attractive sectors. On the contrary, when capital mobility is low, i.e. p is high, capital accumulates at sector $\hat{X}$.

As a consequence, the particular points $\hat{X}_M$ such that $p = 0$ are relatively capital-deterrent sectors.

8.2.2 General remarks

Note that the parameter $p = \frac{M - A(\hat{X})}{f(\hat{X})}$ is a relative version of parameter defined in (102), $f\left(\hat{X}\right)$ excluded:

$$Y(\hat{X}) = M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)$$

However both parameters have the same interpretation in terms of expected long-term returns: both parameters are local maxima when long-term returns $R\left(K_{\hat{X}}, \hat{X}\right)$ are themselves local maxima[15].

Note also that we can interpret the correction $\frac{\sigma_K^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2\sigma_{\hat{X}}^2 \left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$ in $\frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)}$.

A close inspection of equation (68) shows that this term contains -squared- contributions of short-term returns, $f\left(\hat{X}, K_{\hat{X}}\right)$, and the sector's relative attractivity, $\hat{X}$: $\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)$. These contributions are both proportional to the gradient with respect to $K_{\hat{X}}$.

When this gradient is different from zero, i.e. when an increase in capital may improve either the sector's relative attractivity or short-term returns, the correction $\frac{\sigma_K^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2\sigma_{\hat{X}}^2 \left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$ increases $\frac{A\left(\hat{X}\right)}{f\left(\hat{X}\right)}$, and in turn $K_{\hat{X}}$, in most cases. This reflects the tendency of the whole system to reach possibly stable configurations and thus reduce capital discrepancies between close neighbours. Actually, the derivation of the minimization equation in appendix 3.1.2 shows that the term $F\left(\hat{X}, K_{\hat{X}}\right)$ arises as a backreaction of the whole system with respect to modifications at one point of the thread.

8.2.3 Simplification of equation (89) at particular points

Mathematically, the maximization (84) is an equation on $\hat{X}$ with a set of solutions $\hat{X}_M$ and an associated value of $K_{\hat{X}_M}$. The maximum of $A\left(\hat{X}_M\right)$ is M. For such points, the average capital equation (89) simplifies. Actually, at these points, the parameter p is null: $p = 0$. Consequently, given that:

$$\hat{\Gamma}\left(\frac{1}{2}\right) = \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X, K_{\hat{X}, M}\right)\right)^2}{384 \left|f\left(\hat{X}, K_{\hat{X}, M}\right)\right|^3}\right) \tag{109}$$

---

[14] We give an interpretation of $\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2\sigma_{\hat{X}}^2 \left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$ below.

[15] See discussion following equation (102).

equation (89) at points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$ writes:

$$K_{\hat{X},M}\left|f\left(\hat{X}, K_{\hat{X},M}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X},M}\right)\right\|^2 \simeq \sigma_{\hat{K}}^2 C\left(\bar{p}\right)\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X, K_{\hat{X},M}\right)\right)^2}{384\left|f\left(\hat{X}, K_{\hat{X},M}\right)\right|^3}\right) \qquad (110)$$

This equation shows that $K_{\hat{X},M}$ depends, through $\left\|\Psi\left(\hat{X}, K_{\hat{X},M}\right)\right\|^2$ [16], on the form of $R\left(\hat{X}\right)$ and the return $f\left(\hat{X}, K_{\hat{X},M}\right)$. Given the assumptions on $f\left(\hat{X}, K_{\hat{X}}\right)$, equation (110) has several solutions. Actually, $\left\|\Psi\left(\hat{X}, K_{\hat{X},M}\right)\right\|^2$ is decreasing in $K_{\hat{X}}$ (see (293)). If we assume this is also true for $f\left(\hat{X}, K_{\hat{X}}\right)$, equation (89) has two solutions. We give their forms and perform their expansion below.

8.2.4 Particular solutions for capital

We have seen that equation (89) for $p = 0$, that defines average capital per firm per sector, has in general two solutions. We can find an approximate form for these solutions for some particular values of the parameters. By now we will merely consider a power law for $f\left(\hat{X}\right)$:

$$f\left(\hat{X}\right) \simeq B\left(X\right) K_{\hat{X}}^\alpha \qquad (111)$$

The parameter $B\left(X\right)$ is the productivity in sector X, and equation (111) shows that the return $f\left(\hat{X}\right)$ is increasing in $B\left(X\right)$.

The stable case corresponds to setting $\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D$, i.e. given (59), $K_{\hat{X}}^\alpha << D$. In this case, equation (89) rewrites:

$$DB\left(X\right) K_{\hat{X}}^{\alpha+1} = C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B'\left(X\right)\right)^2}{384\left(B\left(X\right)\right)^3} K_{\hat{X}}^{-\alpha}\right)$$

which has for solution:

$$K_{\hat{X}}^\alpha = \left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{-\frac{\alpha}{\alpha+1}}\exp\left(W_0\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B'\left(X\right)\right)^2 \alpha}{384\left(B\left(X\right)\right)^3 \left(\alpha+1\right)}\left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{\frac{\alpha}{\alpha+1}}\right)\right) \qquad (112)$$

where W_0 is the Lambert W function.

For $B\left(X\right) << 1$, we can check that $K_{\hat{X}}^\alpha$ is increasing with $B\left(X\right)$, i.e. with short-term returns $f\left(\hat{X}\right)$ [17], which confirms the results found in approach one: in the stable case, capital equilibrium increases with short-term returns $f\left(\hat{X}\right)$.

The unstable case corresponds to a higher level of capital. Using (59), it amounts to considering in first approximation that:

$$D - \frac{1}{2}\left(\left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}\right) H^2\left(K_X\right) << 1$$

that is, K_X reaches a value such that:

$$\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2 << 1$$

and the capital is concentrated among a small group of agents. Using a power law for $H^2\left(K_X\right)$:

$$H^2\left(K_X\right) = K_X^\alpha$$

---

[16] See equation (59).
[17] See equation (111).

leads to write the solution (110) at the first order in D:

$$K_X^\alpha \simeq \frac{2D}{\left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X,X)}{H(K_X)}} \tag{113}$$

$$f\left(\hat{X}\right) \simeq B\left(X\right) K_{\hat{X}}^\alpha$$

Including corrections of order $\frac{1}{D}$ to formula (113) yields the approximate solution to (110):

$$
\begin{aligned}
K_{\hat{X}}^\alpha \simeq & \frac{2D}{\left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X,X)}{H(K_X)}} \\
& - \left(\frac{\left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X,X)}{H(K_X)}}{2D}\right)^{\frac{1}{\alpha}} \frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)}{DB\left(X\right)} \\
& \times \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B'\left(X\right)\right)^2}{768 D \left(B\left(X\right)\right)^3}\left(\left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R\left(K_X,X\right)}{H\left(K_X\right)}\right)\right)
\end{aligned}
\tag{114}
$$

The previous (in)stability analysis applies. In the range $B\left(X\right) << 1$, when $f\left(\hat{X}\right)$ increases, or which is equivalent, $B\left(X\right)$ increases, average capital must reduce to preserve the possibility of unstable equilibria. Likewise, equilibrium capital is higher when expected returns $R\left(X\right)$ are minimal. When expected returns increase, the threshold defined by the unstable equilibrium decreases.

8.2.5 Expansion around particular solutions for $p = 0$

To better understand the behaviour of the solutions of equation (89), we expand this equation around the points $\left(\hat{X}, K_{\hat{X},M}\right)$ that solve equation (89). Appendix 3.2.2.2 computes this expansion at the second order around $\hat{X}_M$ and $K_{\hat{X},M}$. We find:

$$
\begin{aligned}
\left(K_{\hat{X}} - K_{\hat{X},M}\right) = & \frac{1}{D}\left(\sigma_X^2 \sigma_{\hat{K}}^2 \frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'\left(X\right) f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\right. \\
& \left. -\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial \hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} - \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial \hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right) \\
& + \frac{1}{D}\frac{b}{2}\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right)
\end{aligned}
\tag{115}
$$

with $\frac{A(\hat{X})}{f(\hat{X})}$ given in formula (108):

$$
D = \left(1 + \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + \frac{\sigma_X^2 \sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X}_M}}
$$

and $b = \left(2 - \ln 2 - \gamma_0\right)$, with γ_0 the Euler-Mascheroni constant.

As in the first approach, $D > 0$ corresponds to a stable equilibrium, and $D < 0$ to an unstable one. The expansion (115) describes the local variations of $K_{\hat{X}}$ in the neighbourhood of the points $K_{\hat{X},M}$. This approximation (115) suffices to understand the role of the parameters of the system.

We consider the case of stable equilibria, i.e. $D > 0$. Note that under unstable equilibria, $D < 0$, the interpretations are inverted, since $K_{\hat{X}}$ is interpreted as a threshold[18].

The equation (115), that expands average capital at sector $\hat{X}_M$, is composed of a first order and a second order contributions.

The first order part in the expansion (115) writes:

$$\frac{1}{D}\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4} - \frac{\frac{\partial f(\hat{X},K_{\hat{X}})}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} - \frac{\frac{\partial\left\|\Psi(\hat{X},K_{\hat{X}})\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)$$

(116)

It represents the variation of equilibrium capital as a function of its position. It is decomposed in three contributions:

For $f'\left(\hat{X}\right) > 0$, the second contribution in (116):

$$-\frac{\frac{\partial f(\hat{X},K_{\hat{X}})}{\partial\hat{X}}\left(\hat{X} - \hat{X}_M\right)}{f\left(\hat{X},K_{\hat{X}}\right)}$$

is positive. It represents the decrease in capital needed to reach equilibrium. Actually, the return is higher at point $\hat{X}$ than at $\hat{X}_M$: a lower capital will yield the same overall return at point $\hat{X}$. On the contrary, the first contribution in (116):

$$\frac{\sigma_X^2\sigma_{\hat{K}}^2\frac{3(f'(\hat{X}))^3 - 2f'(X)f''(\hat{X})|f(\hat{X})|}{120|f(\hat{X})|^4}\left(\hat{X} - \hat{X}_M\right)}{D}$$

describes the "net" variation of capital due to a variation in $f(X)$. When returns are decreasing, i.e. when $f'\left(\hat{X}\right) > 0$ and $f''\left(\hat{X}\right) < 0$, this first contribution has the sign of $f'\left(\hat{X}\right)$. An increase in returns attracts capital.

The third term in (116):

$$-\frac{1}{D}\left(\frac{\frac{\partial\left\|\Psi(\hat{X},K_{\hat{X}})\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)$$

represents the number effect. Actually, when:

$$\frac{\frac{\partial\left\|\Psi(\hat{X},K_{\hat{X}})\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} > 0$$

the number of agents is higher at $\hat{X}$ than at $\hat{X}_M$: the average capital per agent is reduced.

The second order contribution in (115) represents the effect of the neighbouring sector space on each sector. Given the first order condition (107):

$$\nabla_{\hat{X}}^2\left(\frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}} = \left(\frac{\nabla_{\hat{X}}^2\left(M - A\left(\hat{X}\right)\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}$$

and since $A\left(\hat{X}_M\right)$ is a maximum, we have:

$$\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M - A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right) > 0$$

---

[18]See the first approach.

When $f\left(\hat{X}\right)$ is constant, $A\left(\hat{X}_M\right)$ is a local maximum, and $K_{\hat{X}_M}$ is a minimum. To put it differently, $K_{\hat{X}}$ is a decreasing function of $A\left(\hat{X}\right)$. This is in line with the definition of $A\left(\hat{X}\right)$[19], which measures the relative attractiveness of sector $\hat{X}$'s neighbours: the higher $A\left(\hat{X}\right)$, the lower the incentive for capital to stay in sector $\hat{X}$.

8.3 Third approach: solving for standard parameter functions

A third approach computes the approximate solutions for the average capital per firm per sector X, which is given by equation (89). To do so, we choose some general forms for the three parameters functions arising in the definition of the action functional, f, g and $H\left(K_X\right)$, given by equations (64), (65) and (59) respectively. Recall that f defines short-term returns, including dividend and expected long-term price variations. The function g describes investors' mobility in the sector space. The function $H\left(K_X\right)$, which is involved in the background field for firms, describes firms' moves in the sectors space.

Once these parameter functions chosen, the approximate solutions of equation (89) for average capital per firm per sector can be found. The second approach has already shown that this equation has in general several solutions. To find these solutions, we thus examine relevant ranges for K_X, namely $K_X >> 1$, $K_X << 1$ and the intermediate range $\infty > K_X > 1$, and find the solutions for each K_X within these various ranges.

8.3.1 Choice of parameter functions

Our choices for the parameter functions f, g and $H^2\left(K_X\right)$ are the following.

Function $H^2\left(K_X\right)$ We can choose for $H^2\left(K_X\right)$ a power function of K_X, so that equation (59) rewrites:

$$\left\|\Psi\left(X\right)\right\|^2 \simeq \frac{D\left(\left\|\Psi\right\|^2\right) - \frac{F}{2\sigma_X^2}\left(\left(\nabla_X R\left(X\right)\right)^2 + \frac{2\sigma_X^2 \nabla_X^2 R\left(K_X,X\right)}{H\left(K_X\right)}\right) K_X^\eta}{2\tau} \equiv D - L\left(X\right)\left(\nabla_X R\left(X\right)\right)^2 K_X^\eta \quad (117)$$

Function f To determine the function f, we must first assume a form for $r\left(K, X\right)$, the physical capital marginal returns, and for F_1, the function that measures the impact of expected long-term return on investment choices.

Assuming the production functions are of Cobb-Douglas type, i.e. $B\left(X\right)K^\alpha$ with $B\left(X\right)$ a productivity factor, we have for $r\left(K, X\right)$:

$$r\left(K, X\right) = \frac{\partial r\left(K, X\right)}{\partial K} = \alpha B\left(X\right) K^{\alpha-1} \quad (118)$$

For function F_1, the simplest choice would be a linear form:

$$F_1\left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\right) \simeq F_1\left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\left\langle K_{\hat{X}}^\alpha\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right) = b\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\left\langle K_X^\alpha\right\rangle\left\langle R\left(X\right)\right\rangle} - 1\right)$$

where, for any function $u\left(\hat{X}\right)$, $\left\langle u\left(\hat{X}\right)\right\rangle$ denotes its average over the sector space, and b an arbitrary parameter.

However, when capital $K_{\hat{X}}^\alpha \to \infty$ and is concentrated at $\hat{X}$, we have $\left\langle K_X^\alpha\right\rangle \simeq \frac{K_{\hat{X}}^\alpha}{N^\alpha(X)}$, so that $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\left\langle K_X^\alpha\right\rangle\left\langle R(X)\right\rangle} \to \frac{N^\alpha(X)R(\hat{X})}{\left\langle R(X)\right\rangle} >> 1$. To impose some bound on moves in the sector space we rather choose:

$$F_1\left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\left\langle K_{\hat{X}}^\alpha\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right) \simeq b\arctan\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\left\langle K_X^\alpha\right\rangle\left\langle R\left(X\right)\right\rangle} - 1\right) \quad (119)$$

---

[19] See discussions after equations (84) and (106).

so that $F_1\left(\frac{R(K_{\hat{X}},\hat{X})}{\langle K_{\hat{X}}^\alpha\rangle\langle R(\hat{X})\rangle}\right) > 0$ when $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_{\hat{X}}^\alpha\rangle\langle R(X)\rangle} > 1$.

Given the above assumptions, the general formula for f given in equation (64) rewrites:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(r\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1} - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + b\arctan\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\langle K_X^\alpha\rangle\langle R(X)\rangle} - 1\right)\right) \tag{120}$$

This general formula can be approximated for $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_X^\alpha\rangle\langle R(X)\rangle} \simeq 1$, when average capital in sector $\hat{X}$ is close to the average capital of the whole space, which is usually the case.

Using our choices (117), (118) and (119) for $\|\Psi(X)\|^2$ $r\left(\hat{X}\right)$ and F_1 respectively, the equation (64) for $f\left(\hat{X},\Psi,\hat{\Psi}\right)$ becomes:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(\left(r\left(\hat{X}\right) + \frac{bR\left(\hat{X}\right)K_{\hat{X}}^\alpha}{\langle K_{\hat{X}}^\alpha\rangle\langle R\left(\hat{X}\right)\rangle}\right) + \gamma L\left(\hat{X}\right)K_X^\eta - \gamma D - b\right)$$

We may assume without impairing the results that $\eta = \alpha$. We thus have:

$$\begin{aligned}
f\left(\hat{X},\Psi,\hat{\Psi}\right) &= \frac{1}{\varepsilon}\left(\left(\frac{r\left(\hat{X}\right)}{K_{\hat{X}}^\alpha} + \frac{bR\left(\hat{X}\right)}{\langle K_{\hat{X}}^\alpha\rangle\langle R\left(\hat{X}\right)\rangle} + \gamma L\left(\hat{X}\right)\right)K_{\hat{X}}^\alpha - \gamma D - b\right) \tag{121}\\
&\equiv B_1\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1} + B_2\left(\hat{X}\right)K_{\hat{X}}^\alpha - C\left(\hat{X}\right)
\end{aligned}$$

where:

$$\begin{aligned}
B_1\left(\hat{X}\right) &= \frac{\alpha B\left(\hat{X}\right)}{\varepsilon}\\
B_2\left(\hat{X}\right) &= \frac{bR\left(\hat{X}\right)}{\varepsilon\langle K_{\hat{X}}^\alpha\rangle\langle R\left(\hat{X}\right)\rangle} + \frac{\gamma}{\varepsilon}\\
C\left(\hat{X}\right) &= \gamma D + b
\end{aligned}$$

Function g To determine the form of function g, equation (65), we must first choose a form for the function F_0.

We assume that:
$$F_0\left(R\left(\hat{X},K_{\hat{X}}\right)\right) = a\arctan\left(K_{\hat{X}}^\alpha R\left(\hat{X}\right)\right) \tag{122}$$

where is a an arbitrary constant.

Combined to our assumption for F_1, (119), the formula (65) for g can be written:

$$g\left(\hat{X},\Psi,\hat{\Psi}\right) = a\nabla_{\hat{X}}\arctan\left(K_{\hat{X}}^\alpha R\left(\hat{X}\right)\right) + b\nabla_{\hat{X}}\arctan\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\langle K_X^\alpha\rangle\langle R(X)\rangle} - 1\right) \tag{123}$$

where the arctan function ensures that the velocity in the sector space g increases with capital and is maximal when average capital per firm in sector $\hat{X}$ tends to infinity, i.e. $K_{\hat{X}}^\alpha \to \infty$.

This general formula, equation (123), can be approximated for $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_X^\alpha\rangle\langle R(X)\rangle} \simeq 1$, when average capital in sector $\hat{X}$ is close to the average capital of the whole space. It then reduces to:

$$g\left(\hat{X},\Psi,\hat{\Psi}\right) \simeq \frac{K_{\hat{X}}^\alpha}{\langle K_{\hat{X}}^\alpha\rangle}\nabla_{\hat{X}}R\left(\hat{X}\right)\left(1 + \frac{b}{\langle R\left(\hat{X}\right)\rangle}\right) \equiv \nabla_{\hat{X}}R\left(\hat{X}\right)A\left(\hat{X}\right)K_{\hat{X}}^\alpha \tag{124}$$

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which in turn allows to approximate the gradient of g, $\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right)$, by:

$$\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right) \simeq \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\left\langle K_{\hat{X}}^\alpha \right\rangle} \left(1 + \frac{b}{\left\langle K_{\hat{X}}^\alpha \right\rangle \left\langle R\left(\hat{X}\right) \right\rangle}\right) K_{\hat{X}}^\alpha \equiv \nabla_{\hat{X}}^2 R\left(\hat{X}\right) A\left(\hat{X}\right) K_{\hat{X}}^\alpha \tag{125}$$

8.3.2   Solutions for the average capital

Now that the forms of the particular functions have been chosen, we can find the approximate solutions to (89) in the ranges of average capital : $K_X \gg 1$, $K_X \ggg 1$, $K_X < 1$ and the intermediate case $\infty > K_X > 1$.

Besides, we only consider the case in which returns are positive[20], $f > 0$.

The asymptotic form of equation (89) which determines average capital per firm per sector, writes:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \left| f\left(\hat{X}\right) \right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96 \left| f\left(\hat{X}\right) \right|^3}\right) \Gamma\left(p+\frac{3}{2}\right) \tag{126}$$

where:

$$p \simeq \frac{M - \left(\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2 \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{\sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}} \tag{127}$$

This equation (126) has several solutions depending on the range of average capital considered.

Case 1: $K_{\hat{X}} \gg 1$ Under the assumption average capital $K_{\hat{X}}$ is large, but not excessively so, we can assume in first approximation that:

$$\left\| \Psi\left(\hat{X}\right) \right\|^2 \simeq D \tag{128}$$

From the expressions (121) and (123), for $f\left(\hat{X}, \Psi, \hat{\Psi}\right)$ and $g\left(\hat{X}, \Psi, \hat{\Psi}\right)$ respectively, we know that $f\left(\hat{X}\right)$ is independent of $K_{\hat{X}}$, and that $g\left(\hat{X}\right)$ is proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$. These functions rewrite:

$$f\left(\hat{X}\right) \equiv c - \frac{d}{K_{\hat{X}}^\alpha R\left(\hat{X}\right)} - \gamma D \tag{129}$$

$$f'\left(\hat{X}\right) \simeq \frac{d \nabla_{\hat{X}} R\left(\hat{X}\right)}{K_{\hat{X}}^\alpha R^2\left(\hat{X}\right)}$$

and:

$$g\left(\hat{X}\right) \simeq -\frac{\nabla_{\hat{X}} R\left(\hat{X}\right) f}{K_{\hat{X}}^\alpha R\left(\hat{X}\right)} \tag{130}$$

$$\nabla_{\hat{X}} g\left(\hat{X}\right) \simeq -\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right) f}{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}$$

We can moreover include the constant α in the definition of $C\left(\bar{p}\right)$, so that equation (126) becomes:

$$K_{\hat{X}} D \left| f\left(\hat{X}\right) \right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96 \left| f\left(\hat{X}\right) \right|^3}\right) \Gamma\left(p+\frac{3}{2}\right) \tag{131}$$

---

[20]Solutions for negative returns, $f < 0$, are discussed below.

Appendix 3.2.3.2 shows that the solution of (??) is:

$$K_{\hat{X}}^{\alpha} = \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{D\left(c-\gamma D\right)} + \frac{d}{\left(c-\gamma D\right)R\left(\hat{X}\right)}\left(1 + M\operatorname{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right)\right) \tag{132}$$

where the function Psi is defined as $\operatorname{Psi}\left(x\right) = \frac{\Gamma'(x)}{\Gamma(x)}$. This solution holds for $K_{\hat{X}} >> 1$, but only when $\left(c-\gamma D\right) > 0$ and $\frac{C(\bar{p})\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{D(c-\gamma D)} >> 1$. Formula (132) shows that the dependency of $K_{\hat{X}}^{\alpha}$ in $R\left(\hat{X}\right)$ in turns depends on the sign of the last term $1 + M\operatorname{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)}\right)$.

When:

$$1 + M\operatorname{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right) > 0$$

average capital in sector $\hat{X}$, $K_{\hat{X}}^{\alpha}$, is decreasing in $R\left(\hat{X}\right)$ and $f\left(\hat{X}\right) \simeq c - \gamma D$. Given the results of section 8.1.3 about stability, this implies that the equilibrium is unstable.

When:

$$1 + M\operatorname{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right) < 0$$

a stable equilibrium, i.e. $K_{\hat{X}}^{\alpha}$ increasing with $R\left(\hat{X}\right)$ and $f\left(\hat{X}\right)$ is possible. This arises for $\nabla_{\hat{X}}^2 R\left(\hat{X}\right) << 0$, for instance when $R\left(\hat{X}\right)$ is maximum. In such a case, an increase in $R\left(\hat{X}\right)$ induces a higher number $\left\|\Psi\left(\hat{X}\right)\right\|^2$ of firms, without impairing average capital per firm.

Case 2: $K_{\hat{X}} >>> 1$ When average capital per firm in sector $\hat{X}$ is very high, and higher than case 1, $K_{\hat{X}} >>> 1$, factor $L\left(\hat{X}\right)$ can be discarded and we can assume that, in first approximation:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D - \left(\left(\nabla_X R\left(X\right)\right)^2 + \frac{2\sigma_X^2 \nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}\right)K_{\hat{X}}^{\alpha} << 1 \tag{133}$$

The function $f\left(\hat{X}, \Psi, \hat{\Psi}\right)$ given in expression (121) can then be rewritten:

$$f\left(\hat{X}\right) \simeq b\left(\frac{\pi}{2} - \frac{\langle K_X^{\alpha}\rangle\langle R\left(X\right)\rangle}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}\right) \equiv c - \frac{d}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} \simeq c$$

The expressions found in the first case, $K_{\hat{X}} >> 1$, for $f'\left(\hat{X}\right)$, $g\left(\hat{X}\right)$ and $\nabla_{\hat{X}}g\left(\hat{X}\right)$, equations (129) and (130) respectively, are still valid.

Appendix 3.2.3.2 solves equation (126) given these assumptions. Two cases arise:

For $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 \neq 0$, we find:

$$K_{\hat{X}}^{\alpha} \simeq \frac{D}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2} - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}}c}$$
$$-\frac{d}{R\left(\hat{X}\right)}\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C\left(\bar{p}\right)\sigma_{\hat{K}}^2\left(\sqrt{\frac{M-c}{c}} + \frac{\frac{M}{c}+\nabla_{\hat{X}}^2 R(\hat{X})\frac{f}{d}}{2\sqrt{\frac{M-c}{c}}}\right)}{c^2 D^{1+\frac{1}{\alpha}}\left(1 - \frac{\left(\nabla_{\hat{X}}R(\hat{X})\right)^{\frac{2}{\alpha}} C(\bar{p})\sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}}\sqrt{\frac{M-c}{c}}\right)} \tag{134}$$

which shows that $K_{\hat{X}}^{\alpha}$ is increasing in $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$ for $K_{\hat{X}}^{\alpha}$ large $\sqrt{f^2\left(\hat{X}\right)} \simeq c << 1$ and $D >> 1$. As explained in section 8.1.3, this corresponds to a stable local equilibrium.

For $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 \rightarrow 0$, we must come back to (117), and replace:

$$(\nabla_X R(X))^2 \rightarrow (\nabla_X R(X))^2 + \frac{2\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)}$$

There are two possibilities:

When $\nabla_X^2 R(K_X, X) < 0$, average capital is given by:

$$K_{\hat{X}} = \left(\frac{C(\bar{p})\sigma_{\hat{K}}^2}{\left|\nabla_X^2 R(K_X, X)\right| c} \Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{c}\right)\right)^{\frac{2}{3\alpha}}$$

This case is unstable. Actually, K_X is decreasing in c, i.e. in $f\left(\hat{X}\right)$. When returns increase, an equilibrium arises only for a relatively low average capital. Otherwise, capital tends to accumulate infinitely. When the sector's expected returns are at a local maximum, the pattern of accumulation becomes unstable. Note that an equilibrium with $K_{\hat{X}} >>> 1$ is merely possible for $c << 1$. Otherwise, there is no equilibrium for $R(K_X, X)$ maximum.

When $\nabla_X^2 R(K_X, X) > 0$, average capital is given by:

$$K_{\hat{X}}^{\alpha} \simeq \left(\frac{D}{\sigma_X^2 \nabla_X^2 R(K_X, X)}\right)^2$$

and points such that $\nabla_X^2 R(K_X, X) > 0$ and $\nabla_X R(X) = 0$ are minima of $R(X)$. This equilibrium may exist only when capital (323) is sufficiently high to compensate for the low expected returns, i.e. to match the condition:

$$\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\langle K_X^{\alpha} \rangle \langle R(X) \rangle} - 1 > 0$$

This equilibrium is thus unlikely and may be discarded in general.

Case 3: $K_{\hat{X}} << 1$ When average physical capital per firm in sector $\hat{X}$ is very low, we can use our assumptions about $g\left(\hat{X}, \Psi, \hat{\Psi}\right)$ and $\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right)$, equations (124) and (125), and assume that:

$$f\left(\hat{X}\right) \simeq B_1\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} >> 1 \tag{135}$$

and:

$$g\left(\hat{X}\right) \simeq 0$$

and moreover that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 = D - L(X)(\nabla_X R(X))^2 K_{\hat{X}}^{\alpha} \simeq D$$

For these conditions, the solution of (126) is locally stable.

Moreover, the conditions $K_{\hat{X}} << 1$ and the defining equation (121) for f imply that $f > 0$, and that for $\alpha < 1$:

$$\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 (f'(X))^2}{96 \left|f\left(\hat{X}\right)\right|^3} << 1$$

Under these assumptions, equation (126) reduces to:

$$K_{\hat{X}} D \left|f\left(\hat{X}\right)\right| \simeq C(\bar{p})\sigma_{\hat{K}}^2 \hat{\Gamma}\left(p + \frac{1}{2}\right) \tag{136}$$

40

This equation (136) can be approximated. Actually, using formula (127) for p yields:

$$p + \frac{1}{2} = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right)}{\sqrt{f^2\left(\hat{X}\right)}} - 1 \simeq -1$$

and an expansion of $\hat{\Gamma}\left(p + \frac{1}{2}\right)$ around the value $p + \frac{1}{2} = -1$ writes:

$$\hat{\Gamma}\left(p + \frac{1}{2}\right) \simeq \hat{\Gamma}(-1) + \hat{\Gamma}'(-1) \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right)}{\sqrt{f^2\left(\hat{X}\right)}}$$

As a consequence, when returns are large, i.e. $f\left(\hat{X}\right) >> 1$, equation (126) writes:

$$K_{\hat{X}} \left(B_1\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} \right) \simeq \frac{C\left(\bar{p}\right) \sigma_{\hat{K}}^2}{D} \left(\hat{\Gamma}(-1) + \hat{\Gamma}'(-1) \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right)}{B_1\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1}} \right)$$

with first order solution[21]:

$$K_{\hat{X}} = \left(\frac{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \hat{\Gamma}(-1)}{DB_1\left(\hat{X}\right)} \right)^{\frac{1}{\alpha}} + \frac{\frac{C(\bar{p})\sigma_{\hat{K}}^2}{D} \hat{\Gamma}'(-1) \left(M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \right)}{B_1^{\frac{1}{\alpha}}\left(\hat{X}\right) \left(\frac{C(\bar{p})\sigma_{\hat{K}}^2 \hat{\Gamma}(-1)}{D} \right)^{1-\frac{1}{\alpha}}} \tag{137}$$

Equation (137) shows that average capital $K_{\hat{X}}$ increases with $M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right)$: when expected long-term returns increase, more capital is allocated to the sector. Equation (135) also shows that average capital $K_{\hat{X}}$ is maximal when returns $R\left(\hat{X}\right)$ are at a local maximum, i.e. when $\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} = 0$ and $\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) < 0$.

Inversely, the same equations (137) and (135) show that average capital $K_{\hat{X}}$ is decreasing in $f\left(\hat{X}\right)$. The equilibrium is unstable. When average capital is very low, i.e. $K_{\hat{X}} << 1$, which is the case studied here, marginal returns are high. Any increase in capital above the threshold widely increases returns, which drives capital towards the next stable equilibrium, with higher $K_{\hat{X}}$. Recall that in this unstable equilibrium, $K_{\hat{X}}$ must be seen as a threshold. The rise in $f\left(\hat{X}\right)$ reduces the threshold $K_{\hat{X}}$, which favours capital accumulation and increases the average capital $K_{\hat{X}}$.

This case is thus an exception: the dependency of $K_{\hat{X}}$ in $R\left(\hat{X}\right)$ is stable, but the dependency in $f\left(\hat{X}\right)$ is unstable. This saddle path type of instability may lead the sector, either towards a higher level of capital (case 4 below) or towards 0. where the sector disappears.

Case 4: $\infty >> K_{\hat{X}} > 1$ intermediate case To solve equation (126) in this general case, we define an intermediate variable W given by:

$$W = \sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2}{96 \left| f\left(\hat{X}\right) \right|^3}} \left(p + \frac{1}{2} - \frac{48 \left| f\left(\hat{X}\right) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2} \left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1 \right) \right) \tag{138}$$

---

[21] Given our hypotheses, $D >> 1$, which implies that $K_{\hat{X}} << 1$, as needed.

so that equation (126) rewrites:

$$
K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \left(\frac{\sigma_X^2 \left(f'(X)\right)^2 \left| f\left(\hat{X}\right) \right| \exp\left(-\frac{96\left| f(\hat{X}) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2 \right)}{96 \left(\sigma_{\hat{K}}^2\right)^3} \right)^{\frac{1}{4}}
$$
$$
= \ C\left(\bar{p}\right) \exp\left(-W^2\right) \sqrt{ W + 2\sqrt{ \frac{96 \left| f\left(\hat{X}\right) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'(X)\right)^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right) } }
$$
(139)

where $\bar{p}$, the average value of parameter p, is defined by $\bar{p} = \bar{p}\left(-M\right)$, and is given by equation (76) when $\lambda = -M$.

Equation (139) can be solved for $K_{\hat{X}}$ under the following simplifying assumptions:

$$
f\left(\hat{X}\right) \simeq B_2\left(X\right) K_{\hat{X}}^{\alpha}
$$
(140)

and:

$$
\left\| \Psi\left(\hat{X}\right) \right\|^2 \simeq D
$$

Eventually, appendix 3.2.3.2 shows that for $\sigma_X^2 << 1$:

$$
K_{\hat{X}}^{\alpha} = \left(\frac{8C\left(\bar{p}\right)}{D} \sqrt{ \frac{3\sigma_{\hat{K}}^2 \left| B_2\left(X\right)\right|}{\sigma_X^2 \left(B_2'(X)\right)^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right) } \right)^{\frac{2\alpha}{1+\alpha}}
$$
(141)
$$
\times \exp\left(-W_0\left(-\frac{48\alpha}{1+\alpha} \left(\sqrt{ \frac{3\sigma_{\hat{K}}^2}{\sigma_X^2} } \frac{8C\left(\bar{p}\right)}{D} \right)^{\frac{2\alpha}{1+\alpha}} \frac{\left| B_2\left(X\right)\right|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X)\right)^{2+\frac{2\alpha}{1+\alpha}}} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2+\frac{\alpha}{1+\alpha}} \right) \right)
$$

where W_0 is the Lambert W function.

In first approximation, equation (141) implies that $K_{\hat{X}}^{\alpha}$ is an increasing function of $B_2\left(X\right)$. Given our simplifying assumption (140), average capital is higher in high short-term returns sectors.

Moreover, $K_{\hat{X}}^{\alpha}$ is a decreasing function of $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$: capital accumulation is locally maximal when expected returns $R\left(\hat{X}\right)$ of sector $\hat{X}$ are at a local maxima, i.e. $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 = 0$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right) < 0$.

Thus, in the intermediate case, the average values $K_{\hat{X}}$ are stable. In addition, both short-term and long term returns matter in the intermediate range.

8.3.3 Case $f < 0$

In the four cases above, we have only considered the case where a sector $\hat{X}$ short-term returns are positive $f\left(\hat{X}\right) > 0$. We can nonetheless extend our analysis to the case $f\left(\hat{X}\right) < 0$.

In such a case, the equilibria, whether stable or unstable, defined in cases 1, 2 with $K_{\hat{X}} >> 1$, and 4 with $K_{\hat{X}} > 1$, are still valid, and capital allocation relies on expectations of high long-term returns. If we consider that $f\left(\hat{X}\right) < 0$ is an extreme case, where expectations of large future profits must offset short-term losses. However, such equilibria become unsustainable when $R\left(\hat{X}\right)$ decreases to such an extent that it does not compensate for the loss $f\left(\hat{X}\right)$. Case 3, $K_{\hat{X}} < 1$ is the only case that is no longer possible when $f\left(\hat{X}\right) < 0$, since the returns that matter in this case are dividends. If they turn negative, the equilibrium is no longer sustainable.

9 Finding average capital in a dynamic environment

So far, we have determined and studied the dependency in parameters of average capital per firm and per sector. However parameters may vary over time, and so should average capital values. We thus introduce a macro time scale and design a dynamic model that involves average capital and time varying expectations in long-term returns.

9.1 Average capital and long-run expected returns

We consider the dynamics for $K_{\hat{X}}$ generated by modifications in parameters. Assuming that some time-dependent parameters modify expected long-term returns $R(X)$, average capital $K_{\hat{X}}$ becomes a function of the time variable θ. To find the evolution over time of the average physical capital per firm in sector $\hat{X}$, $K_{\hat{X}}$, can be found by defining the equation for $K_{\hat{X}}$, (89), and compute its variation with respect to θ, using the fact that the functions $\left\| \Psi\left(\hat{X}\right) \right\|^2$ and $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ both depend on time θ through $K_{\hat{X}}$ and $R(X)$. The variations of these two functions with respect to the two dynamical variables $K_{\hat{X}}$ and $R(X)$ are computed in appendix 4.1. We show that, when $C\left(\bar{p}\right)$ constant, the variation of (89) writes:

$$k\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + l\frac{\nabla_\theta R\left(\hat{X}\right)}{R\left(\hat{X}\right)} - 2m\frac{\nabla_{\hat{X}}\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}} R\left(\hat{X}\right)} + n\frac{\nabla_{\hat{X}}^2 \nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} = -C_3\left(p,\hat{X}\right)\frac{\nabla_\theta r\left(\hat{X}\right)}{f\left(\hat{X}\right)} \tag{142}$$

with:

$$
\begin{aligned}
k &= 1 - \eta\left(1 - \frac{\gamma C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{D^{\frac{1}{\alpha}}c}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}} \\
&\quad + \frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right) - (1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\
l &= \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)} \\
m &= \left(1 - \frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} - \frac{g^2\left(\hat{X}\right)C_2\left(p,\hat{X}\right)}{\sigma_{\hat{X}}^2} \\
n &= \frac{\nabla_{\hat{X}} g\left(\hat{X}\right)C_2\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}
\end{aligned}
\tag{143}
$$

and:

$$
\begin{aligned}
C_1\left(p,\hat{X}\right) &= \frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3} \\
C_2\left(p,\hat{X}\right) &= \ln\left(p+\frac{1}{2}\right) - \frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}} \\
C_3\left(p,\hat{X}\right) &= 1 - C_1\left(p,\hat{X}\right) + \left(p+\frac{3}{2}\right)C_2\left(p,\hat{X}\right)
\end{aligned}
$$

To make the system self-consistent, and since $K_{\hat{X}}$ already depends on R, we merely need to introduce an endogenous dynamics for R.

To do so, we assume that R depends on $K_{\hat{X}}, \hat{X}$ and $\nabla_\theta K_{\hat{X}}$, and that this dependency has the form of a diffusion process (see appendix 4.2). This leads to write R as a function $R\left(K_{\hat{X}}, \hat{X}, \nabla_\theta K_{\hat{X}}\right)$. The variation of R is of the form:

$$
\begin{aligned}
\nabla_\theta R\left(\theta, \hat{X}\right) = \ & a_0\left(\hat{X}\right)\nabla_\theta K_{\hat{X}} + b\left(\hat{X}\right)\nabla_{\hat{X}}^2 \nabla_\theta K_{\hat{X}} + c\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right) + d\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta K_{\hat{X}}\right) \ (144) \\
& + f\left(\hat{X}\right)\nabla_{\hat{X}}^2\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right) + h\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right) \\
& + u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right) + v\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right)
\end{aligned}
$$

We can also assume that the coefficients in the expansion are slowly varying, since they are obtained by computing averages.

This dynamics corresponds to a diffusion process: expected returns in one sector depend on the variations of capital and returns in neighbouring sectors.

To find the intrinsic dynamics for $K_{\hat{X}}$, we assume that the exogenous variation $\frac{\nabla_\theta r(\hat{X})}{r\left(K_{\hat{X}}, \hat{X}\right)}$ is null, and that the system of equations (142) and (144) yields the dynamics for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R\left(\theta, \hat{X}\right)$. Approximating these dynamics to the first order in derivatives, we find:

$$
\begin{aligned}
0 = \ & \begin{pmatrix} \frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} \\ -a_0\left(\hat{X}\right) & 1 \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} \\
& - \begin{pmatrix} 0 & \frac{2m}{\nabla_{\hat{X}} R(\hat{X})}\nabla_{\hat{X}} \\ a\left(\hat{X}\right)\nabla_{\hat{X}} + c\left(\hat{X}\right)\nabla_\theta & e\left(\hat{X}\right)\nabla_{\hat{X}} + g\left(\hat{X}\right)\nabla_\theta \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} \\
& - \begin{pmatrix} 0 & -\frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})}\nabla_{\hat{X}}^2 \\ d\left(\hat{X}\right)\nabla_\theta^2 + b\left(\hat{X}\right)\nabla_{\hat{X}}^2 + u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta & e\left(\hat{X}\right)\nabla_\theta^2 + f\left(\hat{X}\right)\nabla_{\hat{X}}^2 + v\nabla_{\hat{X}}\nabla_\theta \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix}
\end{aligned}
$$

9.2 Oscillatory solutions

We look for solutions of (145) of the type:

$$
\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\left(\hat{X}\right) \end{pmatrix} = \exp\left(i\Omega\left(\hat{X}\right)\theta + iG\left(\hat{X}\right)\hat{X}\right)\begin{pmatrix} \nabla_\theta K_0 \\ \nabla_\theta R_0 \end{pmatrix} \tag{146}
$$

with slowly varying $G\left(\hat{X}\right)$ and $\Omega\left(\hat{X}\right)$. We are then led to the relation between $\Omega\left(\hat{X}\right)$ and $G\left(\hat{X}\right)$:

$$
\begin{aligned}
0 = \ & \frac{k}{K_{\hat{X}}}\left(1 - ieG - ig\Omega\right) + \left(\frac{l}{R\left(\hat{X}\right)} - i\frac{2m}{\nabla_{\hat{X}} R\left(\hat{X}\right)}G\right)\left(a_0 + iaG + ic\Omega\right) \\
& - \frac{l}{R\left(\hat{X}\right)}\left(d\Omega^2 + bG^2 + u\Omega G\right) + \frac{k}{K_{\hat{X}}}\left(e\Omega^2 + fG^2 + v\Omega G\right)
\end{aligned}
$$

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In the sequel, we will limit ourselves to the first order terms which yields the expression for Ω (see appendix 4.3):

$$\Omega = \frac{\frac{lc}{R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right)}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}$$
$$+i\frac{\frac{lc}{R(\hat{X})}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G^2}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2} \tag{147}$$

Once the frequencies (147) of oscillations found, we can determine their condition of stability. When:

$$\frac{lc}{R\left(\hat{X}\right)}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R\left(\hat{X}\right)}\right) + \frac{4m^2 ca_0}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}G^2 > 0 \tag{148}$$

oscillations are dampened and return to the steady state. Otherwise, oscillations are diverging: the system settles on another steady state, i.e. another background state. Appendix 4.4 studies the condition (148) as a function of the parameter functions $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$, the level of average capital $K_{\hat{X}}$, and the coefficients arising in the expectations formations. The results are presented in the next section.

10 Interpretations of the results

Let us now gather and interpret our results. We will discuss the determinants of capital accumulation, its dependency in the parameters, the several patterns of accumulation and the stability of the system, and detail the dynamic system including endogenized expectations. We present a synthesis of the main results obtained at the end of the section.

10.1 Average capital in a given environment

Capital accumulation and the stability of a configuration both depend on several parameters. We will describe the determinants of capital accumulation, its patterns, its dependency in parameters, before studying the density of firms and investors per sector.

10.1.1 Determinants of Capital accumulation

Average capital in sector $\hat{X}$ is determined short-term returns, $f\left(\hat{X}\right)$, which decompose into dividends and price fluctuations. Average capital is also determined by expected long-term returns, $R\left(\hat{X}\right)$, which encompass the gross prospects of the firm. Short term returns and long-term returns are not fully independent: within short-term returns, price fluctuations are driven by expected long-term returns.

A third determinant of average capital is the environment of sector $\hat{X}$, i.e. on the expected long-term return of neighbouring sectors. Mathematically, this means that the derivatives of the expected long-term return $R\left(\hat{X}\right)$ of a sector, along the sectors space do matter for capital accumulation: average capital is determined by $\nabla_{\hat{X}}R\left(\hat{X}\right)$, $\nabla^2_{\hat{X}}R\left(\hat{X}\right)$ and $\nabla_{\hat{X}}f\left(\hat{X}\right)$. Throughout the derivation of the model, we have seen that the dependency of sector $\hat{X}$'s capital accumulation in neighbouring sectors can be measured by the parameter defined in (102), $Y(\hat{X})$:

$$Y(\hat{X}) = M - \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}} + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma^2_{\hat{K}}F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right) \tag{149}$$

or, alternately, its value normalized by short-term returns, defined in (91), up to a constant:

$$p = \frac{M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}_M}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}_M}\right) - \frac{\sigma_K^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2 f^2\left(\hat{X}\right)} \right)}{f\left(\hat{X}\right)} + \frac{3}{2} \tag{150}$$

From these equations, we can see that both $Y(\hat{X})$ and p depend on the gradients of long-term returns $R\left(\hat{X}\right)$ through the function $g\left(\hat{X}\right)$, capital mobility at sector $\hat{X}$.

This function $g\left(\hat{X}\right)$, which depicts investors' propensity to seek higher returns across sectors, is indeed proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$. The gradient of g, $\nabla_{\hat{X}} g$, is proportional to $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$: it measures the position of the sector relative to its neighbours. At a maximum, $\nabla_{\hat{X}}^2 R\left(\hat{X}\right) < 0$, at a minimum, $\nabla_{\hat{X}}^2 R\left(\hat{X}\right) > 0$.

The last term, $\frac{\sigma_K^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2 f^2\left(\hat{X}\right)}$, involved in the definition of $Y(\hat{X})$ and p is a smoothing factor between neighbours' sectors. It will be discussed below.

The parameters $Y(\hat{X})$ and p defined in (149) and (150) represent the relative attractivity of a sector vis-a-vis its neighbours. The parameter p is normalized by short-term returns. It computes the ratio of relative attractivity to short-term returns and allows to consider these two variables separately.

Both parameters, $Y(\hat{X})$ and p, are maximal when $R\left(\hat{X}\right)$ is maximum (see section 8.2.1 and 8.2.2). More generally, the higher $Y(\hat{X})$ and p, the more attractive is sector $\hat{X}$ relative to its neighbours.

10.1.2 Patterns of capital accumulation

Section 8.2 has showed that, for each sector $\hat{X}$, the equation (89) for average capital per firm per sector has, in general, several solutions: these potential equilibria depend on the parameter-functions $f\left(\hat{X}\right)$, $R\left(\hat{X}\right)$ and $Y\left(\hat{X}\right)$, and on firms' densities $\|\Psi\left(X, K_X\right)\|^2$. Each of these parameters influence eachothers. Three patterns of capital accumulation arise. Some will be deemed stable, others unstable with respect to some variations in parameters, a notion that will be detailed later on.

First pattern: low capital, high short-term returns driven by dividends only. In this pattern, the low level of capital implies that firms' returns rely on short term returns $f\left(\hat{X}\right)$ through dividends rather than on expected long-term returns. Here, for low capital, dividends are driven by a high marginal productivity. Capital accumulation depends mainly on short-term returns $f\left(\hat{X}\right)$. Such sectors are in general stable. Agents are scarce: there is a niche effect. Changes in parameters impact $f\left(\hat{X}\right)$ through a change in marginal returns are adjusted for by a change in the number of firms.

Second pattern: intermediate to high level of capital, short-term returns, long-term expectations. Capital accumulation increases with any parameter that increases short-term returns - dividends and stock prices - or long-term returns through relative attractivity $Y\left(\hat{X}\right)$, as detailed above: locally, the higher the sector relative attractivity, the higher capital accumulation. This is the most standard pattern of capital allocation.

Third pattern: high capital, long-term returns and relative attractivity In these sectors, capital accumulation depends on high expected long-term returns, themselves sustained by high levels of capital. Sectors with maximal expected returns, i.e. maximal attractivity, dominate their neighbours and may accumulate extremely high levels of capital. However, such equilibria are unstable (see section 8.1): in a dynamic perspective, all else equal, capital could grow indefinitely.

Combined, the two last patterns show that, in sectors where expected long-term returns are maximal, two outcomes may arise: a stable pattern of high capital, and an unstable pattern.

A last particular case arise in the third pattern. It is a limit case of our model, where low expected returns do not deter extremely high capital. However this case is unsustainable.

10.1.3 Dependency in parameters

The stability of the configuration influences the dependency in the parameters of the system.

Recall that in the stable case (patterns 1,2 and partly 3), average values can be understood as equilibrium values (see section 8.1), but that unstable equilibria rather describe potential thresholds, not actual ones.

In locally stable configurations, average capital is increasing in short-term returns, expected long-term returns, and relative attractivity of the sector, $f\left(\hat{X}\right)$, $R\left(\hat{X}\right)$ and $Y\left(\hat{X}\right)$ respectively. The higher the returns, the higher the capital accumulation.

For unstable equilibria, on the contrary, average capital is decreasing in these variables: an increase in short-term returns or expected long-term returns facilitates capital accumulation and reduces the threshold. When capital moves below these thresholds, the sector average capital move toward the next stable equilibrium, whereas when capital exceeds the threshold, the sector's average capital per firm grows indefinitely.

This form of instability will be discussed below, and will later lead to consider the dynamics aspect of the average values of capital per sector at macro-time scale.

Lastly, we can compare the relative effect of parameters' variation on neighbouring sectors. An equal increase of long-term returns per unit of capital, $R\left(\hat{X}\right)$, in two close sectors favours the best capitalised sector, since its total returns' expectations are higher. Capital flows thus increase inequality between neighbouring sectors. On the other hand, an increase in global productivity impacts short-turn returns, $f\left(\hat{X}\right)$, of least capitalized sectors and lures in investors.

10.1.4 Density of firms per sector

Formula (59) defines the density of firms per sector. Its central parameter depends on expected long-term returns. For any given level of capital, the number of firms in sector X is given by (59):

$$\|\Psi\left(X\right)\|^2 = (2\tau)^{-1}\left(D\left(\|\Psi\|^2\right) - \frac{1}{2}\left(\left(\nabla_X R\left(X\right)\right)^2 + \frac{\sigma_X^2 \nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}\right)H^2\left(K_X\right)\left(1 - \frac{H'\left(\hat{K}_X\right)K_X}{H\left(\hat{K}_X\right)}\right)\right)$$
(151)

and this function is decreasing in:

$$\left(\nabla_X R\left(X\right)\right)^2 + \frac{\sigma_X^2 \nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}$$

Recall that $\nabla_X R\left(X\right)$ is the gradient of expected long-term returns relative to the sectors space, and that $\nabla_X^2 R\left(K_X, X\right)$ is the Laplacian, i.e. the generalisation of the second derivative of $R\left(K_X, X\right)$ with respect to the sectors' space.

When $\nabla_X R\left(X\right) \neq 0$, the sector is only 'transitory". The sector X is on a slope: neighbouring sectors have lower expected returns, others have higher ones. Firms head towards the sectors with higher returns. The higher the slope $\nabla_X R\left(X\right)$, the faster firms leave the sector.

For $\nabla_X R\left(X\right) = 0$ and $\nabla_X^2 R\left(K_X, X\right) < 0$, sector X's returns $R\left(X\right)$ are at a local maximum. Formula (151) shows that the number of firms in the sector is maximal, since firms tend to accumulate capital in most profitable sectors. Thus, at these points, there are both a large number of firms and high level of capital per firm. Yet competition ensures that some firms do remain in sectors with low, or even minimal expected returns. We will see that these equilibria are unstable.

For $\nabla_X R\left(X\right) = 0$ and $\nabla_X^2 R\left(K_X, X\right) > 0$, the density of firms is much lower than in the case where $\nabla_X^2 R\left(K_X, X\right) < 0$. Potential incomers are crowded out by competitors with higher capital. Competition also implies that, for given returns $R\left(X\right)$, the number of firms in a sector X decreases with the average level of capital K_X. Equilibria in sector X with high level of capital per firm K_X have a relatively low number of firms: firms with high capital deter incomers.

10.1.5 Density of investors per sector

Formula (78) shows that the average number of investors at sector $\hat{X}$ is an increasing function of short-term returns, $f\left(\hat{X}\right)$, and the sector $\hat{X}$'s relative long-term attractivity, p, defined in equation (91). All else equal, an increase in short-term returns or an improvement of the sector's relative long-term attractivity increases the number of investors and, through them, the disposable capital for firms.

The density of investors in sector $\hat{X}$ increases with its relative attractivity p. which can be written as:

$$p = \frac{M - \left(\left(g\left(\hat{X} \right) \right)^2 + \sigma_{\hat{X}}^2 \left(f\left(\hat{X} \right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}} \right) \right) \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X} \right)}} + \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}} \right)}{2\sigma_{\hat{X}}^2 \left(\sqrt{f^2\left(\hat{X} \right)} \right)^3}$$

where the first term is the attractivity of sector's $\hat{X}$ relative to its neighbours, normalized by short-term returns $f\left(\hat{X} \right)$, and the second is a smoothing term that reduces differences between sectors: it increases when the relative attractivity with respect to $K_{\hat{X}}$ decreases. The number of investors and capital will increase in sectors that are in the neighbourhood of significantly more attractive sectors, i.e. with higher average capital and number of investors.

10.2 (In)Stability

Two sources of instability may arise in the model. One is local, and stems from solutions for average capital per firm per sector, equation (89). Another is global, and stems from the constraint imposed, in the model, on the total number of investors.

10.2.1 Local stability of capital accumulation patterns

Mechanisms of local instability   The equation (89) defining average capital per firm per sector can itself be seen as the stable point of a dynamical equation with varying parameters[22] (see (99)):

$$\delta K_{\hat{X}}\left(t+1 \right) = \left(-\left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}} \right)}{\partial K_{\hat{X}}}}{f\left(\hat{X}, K_{\hat{X}} \right)} + \frac{\frac{\partial \left\| \Psi\left(\hat{X}, K_{\hat{X}} \right) \right\|^2}{\partial K_{\hat{X}}}}{\left\| \Psi\left(\hat{X}, K_{\hat{X}} \right) \right\|^2} + l\left(\hat{X}, K_{\hat{X}} \right) \right) + k\left(p \right) \frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}} \delta K_{\hat{X}}\left(t \right)$$

$$+ \frac{\partial}{\partial Y\left(\hat{X}, t \right)} \left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p} \right) 2\hat{\Gamma}\left(p + \frac{1}{2} \right)}{\left| f\left(\hat{X}, K_{\hat{X}} \right) \right| \left\| \Psi\left(\hat{X}, K_{\hat{X}} \right) \right\|^2} \right) \delta Y\left(\hat{X}, t \right) \tag{152}$$

Capital accumulation is potentially instable for some sectors (see (101a)), when:

$$\left| B\left(\hat{X} \right) \right| \equiv \left| k\left(p \right) \frac{\partial p}{\partial K_{\hat{X}}} - \left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}} \right)}{\partial K_{\hat{X}}}}{f\left(\hat{X}, K_{\hat{X}} \right)} + \frac{\frac{\partial \left\| \Psi\left(\hat{X}, K_{\hat{X}} \right) \right\|^2}{\partial K_{\hat{X}}}}{\left\| \Psi\left(\hat{X}, K_{\hat{X}} \right) \right\|^2} + l\left(\hat{X}, K_{\hat{X}} \right) \right) \right| > 1 \tag{153}$$

When this inequality holds, the equilibrium is not a steady state: a variation $\delta Y\left(\hat{X}, t \right)$ drives the system away from potential equilibria. This variation shifts the average capital per firm $K_{\hat{X}}$ via the four terms in $\left| B\left(\hat{X} \right) \right|$:

The first term $\frac{\partial f\left(\hat{X}, K_{\hat{X}} \right)}{\partial K_{\hat{X}}}$ captures the relative variation of short-term returns, dividends and price fluctuations.

The second term is the variation in the number of firms moving in, or out of, sector $K_{\hat{X}}$.

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[22] The definitions of the parameters have been given after equation (94).

The two last term $l\left(\hat{X}, K_{\hat{X}}\right)$ and $k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}$ compute the modification in the background field induced by the indirect change $\delta Y\left(\hat{X}, t\right)$.

The term $l\left(\hat{X}, K_{\hat{X}}\right)$ is the direct variation induced by a modification of $f\left(\hat{X}\right)$.

The term $k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}$ is the variation induced by a modification of $p\left(\hat{X}\right)$, the relative position of sector $\hat{X}$ in the space of return, which depends on the shape of the returns around $\hat{X}$.

Thus, the modification of one parameter affects the system as a whole, and reshapes the collective state through modifications of the background field.

The sum of these four modifications magnifies or dampens any initial modification and, depending on the amplitude of $\left|B\left(\hat{X}\right)\right|$, determines the stability of the system.

Consequences of local instability   Instability can arise in two cases[23]:

When $B\left(\hat{X}\right) < -1$, $K_{\hat{X}} \to 0$ and the short-term return becomes infinite $f\left(\hat{X}\right) \to \infty$. In such a case, the background field becomes null at sector $\hat{X}$, $\left\|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right\|^2 \to 0$, i.e. firms desert the sector**.** This corresponds to the extreme case of a niche effect. Investors and firms are scarce and returns are high, since, for a very low capital marginal productivity is mathematically high. However the total capital involved in this case is negligible, and does not impact the system globally.

When $B\left(\hat{X}\right) > 1$, $K_{\hat{X}} \to \infty$ and $f\left(\hat{X}\right) \to c$ for some constant $c << 1$.

Given the definition of the parameters, we can assume that for $K_{\hat{X}} \to \infty$, $\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}} \to 0$ so that $\frac{\partial p}{\partial K_{\hat{X}}} \to 0$. Moreover, $l\left(\hat{X}, K_{\hat{X}}\right) \to 0$. As a consequence, (153) becomes:

$$\frac{\frac{\partial \left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} > 1$$

which implies that $\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2$ behaves approximatively as:

$$\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2 \simeq \exp\left(rK_{\hat{X}}\right)$$

with $r > 1$.

Since $K_{\hat{X}} \to \infty$, the background field $\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2 \to \infty$ and the number of firms grows indefinitely. The number of agents is fixed: they move towards sector $\hat{X}$, until some maximum number of agents is reached, and:

$$\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2 = \left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2_{\max} >> 1$$

and the capital at sector $\hat{X}$ reaches:

$$K_{\hat{X}} \simeq K_{\max} = \frac{\ln\left(\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2_{\max}\right)}{r}$$

So that when a point exists where $B\left(\hat{X}\right) > 1$, the total remaining capital available $V\left\langle K\right\rangle$ is reduced to $V\left\langle K\right\rangle - K_{\max}$, with V the volume of the sector space and $\left\langle K\right\rangle$ the average physical capital in the space. Average capital available reduces to $\left\langle K\right\rangle - \frac{K_{\max}}{V}$, which directly impacts other points in the sectors' space,

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[23]We assume physical capital returns are Cobb-Douglas.

since capital accumulation in one point depends explicitly on the function:

$$F_1 \left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right) \left\| \Psi\left(X'\right) \right\|^2 dX'} \right)$$

The function F_1 depends on $\langle K \rangle - \frac{K_{\max}}{V}$, which in turn modifies the function $f\left(\hat{X}\right)$ and $\left| B\left(\hat{X}\right) \right|$ over the whole space: some points move over the threshold $B\left(\hat{X}\right) > 1$, others below $B\left(\hat{X}\right) < -1$. Some sectors experience a capital increase, others disappear. As a consequence, if a stable situation finally emerges, the sector space may be considered as a reduced one: some sectors disappear, and only sectors with positive capital remain. Ultimately, the state defined by $\hat{\Psi}\left(\hat{X}, \hat{K}\right)$ should be transformed into a field $\hat{\Psi}_{\mathrm{red}}\left(\hat{X}, \hat{K}\right)$ of stable equilibria on this reduced space: the model should be transformed ultimately in another one.

10.2.2 Global stability

A second source of instability of the system arises outside of the equations for average capital per firm per sector, (89), and its differential version, (93). It stems from the sectors' space expected long-term returns. It is induced by the minimization equations (66) and (67), and is a source of global instability for the background field.

Description of global instability In these equations, the Lagrange multiplier $\hat{\lambda}$ is the eigenvalue of a second-order differential equation. Because there exist an infinite number of eigenvalues $\hat{\lambda}$, there are an infinite number of local minimum background fields $\Psi\left(\hat{X}, K_{\hat{X}}\right)$. But the most likely minimum, given in (84), is obtained for $\hat{\lambda} = M$ (see appendix 2).

Yet $\hat{\lambda}$ is also the Lagrange multiplier that implements the constraint of a fixed number N of agents.

Since the number of investors is computed by:

$$\int \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right) \right\|^2 d\left(\hat{X}, K_{\hat{X}}\right)$$

the constraint implemented by $\hat{\lambda}$ is:

$$\hat{N} = \int \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right) \right\|^2 d\left(\hat{X}, K_{\hat{X}}\right) \tag{154}$$

since this constraint runs over the whole space, it is a global property of the system.

Yet equations (66) and (67), the minimization equations defining $\Psi\left(\hat{X}, K_{\hat{X}}\right)$, may also be viewed as a set of local minimization equations at each point $\hat{X}$ of the sector space. Considered individually, each provide a lower minimum that could be reached separately for each $\hat{X}$. In other words, provided each sector's number of agents is fixed independently from the rest of the system, a stable background field could be reached at every point.

However, our global constraint rules out this set of local minimizations. The solutions of (66) and (67) are thus a local minimum for the sole points $\hat{X}$ such that M: the lowest value of $\hat{\lambda}$ is reached at $\hat{X}$, such that:

$$Y\left(\hat{X}\right) = \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2 f^2\left(\hat{X}\right)} = M \tag{155}$$

For points $\hat{X}$ that do not satisfy (155), the solution $\Psi\left(\hat{X}, K_{\hat{X}}\right)$ and $\Psi^\dagger\left(\hat{X}, K_{\hat{X}}\right)$ of (66) and (67), with $\hat{\lambda} = M$ are not a global minimum, but merely a local one. Any perturbation $\delta\Psi\left(\hat{X}, K_{\hat{X}}\right)$ in the parameters destabilize the system: the equilibrium is unstable.

The stability of both the background field and the potential equilibria are thus determined by $Y\left(\hat{X}\right)$, the sector space's overall shape of returns and expectations. An homogeneous shape, a space such that $Y\left(\hat{X}\right)$, presents small deviations around M and is more background-stable than an heterogeneous space.

More importantly, the background fields and associated average capital must be understood as potential, not actual long-run equilibria: the whole system is better described as a dynamical system, which will be defined in section 9, between potential backgrounds where time enters as a macro-variable. We consider the results of the background field's dynamical behaviour in section 10.3.

Reducing agents mobility and removing instability As mentioned above, an homogeneous shape is a space such that $Y\left(\hat{X}\right)$ presents small deviations around M. In an heterogeneous shape, the space presents large differences in $Y\left(\hat{X}\right)$. We find that homogeneous shapes are more background-stable than heterogeneous ones. This partly results from the global constraint (154) imposed on the number of agents in the model, which ensures that the number of financial agents in the system is fixed over the whole sector space.

Relaxing this constraint fully would render the number of agents in sectors independent. The associated background field of each sector could, at each point, adjust to be minimum and stabilize the system.

To do so, we replace equation (66), the minimization equation, by a set of independent equations with independent Lagrange multipliers $\hat{\lambda}_{\hat{X}}$ for each sector $\hat{X}$:

$$
\begin{aligned}
0 =\ & \left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 - \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right)\right)\hat{\Psi} \\
& + \left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X}, K_{\hat{X}}\right)\right) - \hat{\lambda}_{\hat{X}}\right)\hat{\Psi} - F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\hat{\Psi}
\end{aligned}
\tag{156}
$$

For each $\hat{X}$, the minimum configuration is reached by setting:

$$
\hat{\lambda}_{\hat{X}} = \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}
$$

which is similar to the Lagrange multiplier (84) of the minimization equation for the background field, stripped of the maximum condition, and where the average capital equation (89) is replaced by[24]:

$$
K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right| = C\left(\bar{p}\right)\sigma_{\hat{K}}^2\hat{\Gamma}\left(\frac{1}{2}\right) = C\left(\bar{p}\right)\sigma_{\hat{K}}^2\exp\left(-\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(f'\left(X, K_{\hat{X}}\right)\right)^2}{384\left|f\left(\hat{X}, K_{\hat{X}}\right)\right|^3}\right)
\tag{157}
$$

This equation is identical to (110) and has thus at least one locally stable solution. The solutions are computed in (112) and (114).

Solutions to (157) do no longer directly depend on the relative characteristics of a particular sector, but rather on the returns at point $f\left(\hat{X}\right)$ and on the number of firms in the sector, $\left\|\Psi\left(\hat{X}\right)\right\|^2$. Yet this dependency is only indirect, through the firms' density at sector $\hat{X}$, $\left\|\Psi\left(\hat{X}\right)\right\|^2$, and this quantity does not vary much in the sector space.

An intermediate situation between (89) and (157) could also be considered: it would be to assume a constant number of agents in some regions of the sector space.

Alternatively, limiting the number of investors per sector can be achieved through some public regulation to maintain a constant flow of investment in the sector.

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[24]Expression (109) is used to compute $\hat{\Gamma}\left(\frac{1}{2}\right)$.

10.3 Average capital in a dynamic environment

Recall that we have extended our model in section 9 by endogenizing the expected long-term returns $R\left(\hat{X}\right)$. We have found a dynamic equation (145) for the two variables $\left(K_{\hat{X}}, R\left(\hat{X}\right)\right)$ system, and shown that this system has oscillatory solutions (146):

$$\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\left(\hat{X}\right) \end{pmatrix} = \exp\left(i\Omega\left(\hat{X}\right)\theta + iG\left(\hat{X}\right)\hat{X}\right) \begin{pmatrix} \nabla_\theta K_0 \\ \nabla_\theta R_0 \end{pmatrix} \tag{158}$$

where the frequencies $\Omega\left(\hat{X}\right)$ satisfy (147):

$$\Omega = \frac{\frac{lc}{R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right)}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}$$

$$+i\frac{\frac{lc}{R(\hat{X})}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G^2}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2} \tag{159}$$

The coefficients arising in the frequency equation, given in (143) and (144), depend on $K_{\hat{X}}$ and on the form of the expectations' formation (144).

We will now study the frequency Ω in equation (159) as a function of both average capital $K_{\hat{X}}$ and the form of expectations[25]. We are mainly interested in the imaginary part of (147), whose sign determines whether the dynamics is stable.

We have seen in (148) that the condition for stable oscillations is:

$$\frac{lc}{R\left(\hat{X}\right)}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R\left(\hat{X}\right)}\right) + \frac{4m^2 ca_0}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}G^2 > 0 \tag{160}$$

When this condition is satisfied, oscillations at point $\hat{X}$ dampen and the system returns to its equilibrium, i.e. the background state. Otherwise, diverging oscillations lead the system to the next background state.

To investigate these two possibilities, the most important parameter is the coefficient c in equation (160) that has been defined in equation (144). Its values determine two relevant forms of expectations. Equation (144) shows that, when $c > 0$, expectations are highly reactive to variations in capital, i.e. expected long-term returns depend positively on the variations of average capital $K_{\hat{X}}$. When $c < 0$, expectations are moderately reactive to variations in capital, i.e. expected long-term returns depend negatively on the variations of average capital $K_{\hat{X}}$.

10.3.1 Case 1: $K_{\hat{X}} << 1$

When average capital is very low, for $K_{\hat{X}} << 1$, the dominant coefficient in the equation defining the frequencies of oscillations (147) is $\left|\frac{k}{K_{\hat{X}}}\right| >> l >> 1$, so that the solution of (147) is stable if:

$$\frac{cl}{R\left(\hat{X}\right)}\frac{k}{K_{\hat{X}}} > 0 \tag{161}$$

Appendix 4.4 shows that for $K_{\hat{X}} << 1$ we have $k < 0$. As a consequence, equation (161) implies that, for $K_{\hat{X}} \leqslant 1$ and $c > 0$, i.e. for highly reactive expectations, oscillations are unstable, whereas for $c < 0$, i.e. moderately reactive expectations, they are stable.

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[25] See appendix 3.

This is in line with the notion of stable average capital defined above. Actually, for such stable values, there is a positive relation between variations of $R\left(\hat{X}\right)$ and average capital.

In addition, for $c > 0$, expected long-term returns depend positively on the variations of average capital $K_{\hat{X}}$. This creates an amplification in the dynamics of these two variables.

On the contrary, for $K_{\hat{X}} \leqslant 1$ and $c < 0$, the expected long-term returns depend negatively on the variations of average capital $K_{\hat{X}}$. Stabilization occurs with dampening oscillations. This shows that for expectations moderately reactive to variations of capital, some equilibria with relative low capital are possible and resilient to oscillations in expectations, a niche effect may exist for some sectors.

10.3.2 Case 2: $K_{\hat{X}} >> 1$

For $K_{\hat{X}} >> 1$, on the contrary, both in stable and unstable equilibrium with high average capital, oscillations are dampening for $c > 0$ and explosive for $c < 0$.

When $c > 0$, expectations are highly reactive and this leads to an amplification between variations of capital and return expectations.

In the stable case, high average capital depends on this amplification by expectations: fluctuations that would otherwise be destabilizing for low capital sectors may stabilize or maintain stable sectors with high level of capital. This does not mean that these sectors become attraction points, but rather that a large reactivity between expectations and capital will allow for their intrinsic high level of capital to consolidate.

In the unstable case, due to the specificity of the equilibrium for $K_{\hat{X}} >> 1$, the mechanism of stabilization is as follows: an initial increase in $K_{\hat{X}}$ implies an increase in the expected long-term return. But in turn, the negative relation between $K_{\hat{X}}$ and the variations of $R\left(\hat{X}\right)$ lowers the average value of the average capital. Thus, an increase in equilibrium capital $K_{\hat{X}}$ improves sector $\hat{X}$'s profitability, which in turn lowers the entry's threshold in this sector and ultimately reduces the potential equilibrium level of capital. Put differently, an initial rise in the sector's average capital impacts with amplification the sector's expected return, which reduces the potential average equilibrium by an amount that offsets the initial rise in capital.

For $c < 0$, moderately reactive expectations impair the dampening oscillations mechanism that arise for $c > 0$. For instance in the unstable case, an initial rise in the threshold $K_{\hat{X}}$, impacts moderately sectors' expected returns, without offsetting the initial rise in capital.

10.3.3 Case 3: Intermediate values $\infty > K_{\hat{X}} > 1$

For intermediate values of capital, several possibilities arise.

When $c < 0$, the oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma K_{\hat{X}} F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)} < 0 \tag{162}$$

and:

$$G^2 < \frac{l^2 \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}{4a_0 R\left(\hat{X}\right)} \left(\frac{\varsigma \left(\nabla_{\hat{X}} R\left(\hat{X}\right) F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\gamma \left(D - \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}\right)^2 \left|\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}\right| \tag{163}$$

The constant ς is irrelevant here, although it arises in appendix 3 to estimate short-term returns, and the function F_1, defined in (2), determines the stock's prices evolution. The coefficient α is the Cobb-Douglas power arising in the dividend part of short-term returns. The constant D, defined in (59), determines the relation between number of firms and average capital at sector $\hat{X}$.

Conditions (162) and (163) correspond to the case of relatively low capital for which a stability in the oscillations may be reached when expectations are moderately reactive to variation in capital.

For $c > 0$ the oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma K_{\hat{X}} F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)} > 0$$

or if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma K_{\hat{X}} F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)} < 0$$

and:

$$G^2 > \frac{l^2 \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}{4a_0 R\left(\hat{X}\right)} \left(\frac{\varsigma \left(\nabla_{\hat{X}} R\left(\hat{X}\right) F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\gamma \left(D - \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}\right)^2 \left|\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}\right|$$

We recover the large average capital case. A relatively high reactivity of expectations to fluctuations in capital allows to maintain the capital at its equilibrium value.

10.3.4 two types of oscillations

These results show that the threshold's values between dampening and explosive oscillations depend on the sectors and on the parameters of the system. Interactions between moderately reactive expectations and capital favour patterns 1 and 2, i.e low to high capital sectors, and to impair pattern 3, i.e. very high capital sectors.

On the other hand, very reactive expectations favour pattern 3 and impair patterns 1 and 2. Actually, in this case, oscillations are relatively weaker for high capital sectors, which leads to a reallocation of capital. If, for instance, expected long-term returns decrease in the neighbourhood of a high capital sector, capital will be reallocated from neighbours to the considered sector and will stabilize the high capital sector.

Recall moreover, that extreme cases of pattern 3, i.e. both maximal capital and returns, represent moving thresholds that repulse low-capital firms and allow already high-capital firms to grow indefinitely. These thresholds and their oscillations should generate highly global instability: oscillations constantly drive firms below the threshold out of the sector.

To conclude, let us stress that, as in the static case, the dynamics of average capital and expected returns can itself be seen as a dynamics for the system's background fields or collective states. As a consequence, the background fields are themselves subject to fluctuations. Moreover, since oscillations around a collective state may destabilize the patterns in some sectors, fluctuations may ultimately switch the collective state and modify the repartition of patterns across the sectors.

10.4 Synthesis of the results

Let us now synthetize our results.

1. In our model, when firms reallocate their capital, they tend to do so in sectors with relatively higher long-term returns. The speed at which they reallocate depends on their capital endowment, but can be crowded out by their competitors. The higher the firm's capital, te greater the chance to overcome competitors. Eventually, highest capital firms concentrate in highest expected long-term return sectors, while the rest locate in neighbours sectors, and possibly least expected return sectors.

2. Financial capital allocation depends on short-term returns, dividends and price fluctuations, and expected long-term returns. However, since price fluctuations are driven by expected long-term returns, short and long-term returns are not independent. Financial capital allocation also depends on the sector's relative attractivity, which measure the expected returns of a sector relative to its neighbours. However financial capital is volatile. High short-term returns are an incentive, but the relative attractivity of sectors lures investors. Financial capital allocation thus depends on the ratio of sectors' relative attractivity to short-term returns. Since this ratio depends on expectations, it is subject to fluctuations, which in turn impact the collective state.

3. Capital allocation of firms and investors differ and interact. They also impact the form of the background field and the average values of capital per sector. Average capital per firm per sector depends on short-term returns, both dividends, that are driven by marginal productivity, and stock prices fluctuations, and on expectations of long-term returns.

4. For each sector, three patterns of accumulation emerge. They depend on three parameters of the model, short-term returns, expected long-term returns and the sector's relative attractivity. In the first pattern, the dividend component of short-term returns is determinant for sectors with small number of firms and low capital. In the second pattern, both short and long-term returns drive intermediate-to-high capital in the sector. In the third pattern, higher expectations of long-term returns drive massive inputs of capital in sectors. In this pattern, firms with maximal expected return can theoretically accumulate without bound. Practically this accumulation stops when there is no more capital available.

5. These values of average capital are stationary results: agents move and accumulate but, in average, density of firms and average capital per sector are constant.

6. In a dynamic perspective, the patterns of accumulation are in general stable: small deviations from the average values induce the system to return to its initial values. Only sectors in the third pattern with maximal expected returns are unstable. The average value of capital per firm located at these sectors is a threshold: firms with capital below this threshold move toward the next stable equilibrium, firms with capital above accumulate indefinitely.

7. The results above, for average values of capital, hold for a given background field. However, for each sector, three possible patterns of capital accumulation may exist. This combination of various accumulation patterns in each sector yields an infinite number of possible collective states. It does not follow that these various possibilities are free: sector patterns depend on the relative attractivity of both the sector and its neighbours'. There are also constraints: massive inflows of capital are only driven by high expected long-term returns, while niche effects merely occur for relatively high productivity firms. However, from relatively homogeneous levels of capital to largely heterogeneous patterns of accumulation between sectors, a potentially infinite range of collective states may exist. So that when parameters vary, given collective states may switch to another: a change in expectations or parameters may, for instance, induce variations in average capital, and in turn induce changes in sectors' patterns of capital accumulation. To study these possible switches, we introduce a dynamic interaction between average capital and expected long-term returns, now endogenized.

8. The main characteristic of this dynamic interaction depends both on the patterns of accumulation and expectations formation. Two types of expectations are relevant: expectations of long-term returns, that react positively to any variation in capital, i. e. highly reactive expectations, and expectations reacting negatively to this variation, i.e. moderately reactive. In this dynamics, average capital and expectations present some oscillatory patterns that may dampen equilibria or drive them towards other equilibria. Expectations highly reactive to capital variations stabilize high-capital configurations. They drive low-to-moderate capital sectors towards zero or higher capital, depending on their initial conditions. Inversely, expectations moderately reactive to capital variations stabilize low-to moderate capital configurations, and drive high-capital sectors towards lower capital equilibria.

11 Discussion

The use of statistical field theory has led us to describe a microeconomic framework in terms of collective states. These collective states are composed of sectors themselves made up of a large number of firms. Recall that our notion of firm is versatile: a single company could be modelled as a set of independent firms. Similarly, the notion of sector merely refers to a group of entities with similar activities.

Each collective state is singularly determined by the collection of data that characterizes each sector: number of firms, number of investors, average capital and density of distribution of capital. Mathematically, the collective state is described by what we have called a background field. Any change in this collection of data would describe another collective state. However collective states do not change at the slightest variation of one of these data: they deal with theoretical averages over long-term periods, not instantaneous empirical averages. Nor are collective states arbitrary: they directly result from agents' interactions and are the most probable stable states of the system obtained by minimization conditions. Other states exist, but they are unstable.

The collective states describe the possible background states of the economy considered that eventually condition the agents' individual dynamics. They depend on the parameters of the model, short-term and long-term returns, relative attractiveness of the sector, and any parameter conditioning these three quantities.

Their multiplicity stems from the multiple possibility of patterns in each sector. For instance, pattern-3, stable and unstable, are more present in the US than in the UK stock markets, where pattern 1 and 2 dominate.

A particular collective state can be described by its distribution into patterns of capital accumulation - type 1, 2 or 3 - across sectors. Each sector has its own pattern of accumulation, and the distribution in patterns is directly conditioned by the economic constraints imposed on the system. Type-3 patterns appear in sectors locally more attractive in the long-term. It is this relative attractivity that determines the sector's capital. Patterns 1 or 2, that are relatively less attractive sectors, lure in capital with dividends and expected returns. Besides, sectors are connected and benefit from the relative attractiveness of their neighbours: this smoothing effect between sectors materialise in mergers and acquisitions.

The selection of a particular collective state and its sectoral patterns is ultimately determined by exogenous conditions. Structural changes, such as an extra-loose monetary policy or the choice of a pension system are external conditions that modify collective states.

Collective states are not static. Their dynamics depend on short-term and long-term return functions, that are exogenous, and more broadly on a whole landscape of technological and economic conditions. But as a system, they also present an internal dynamics. We have considered these two types of variations in the paper.

First, exogeneous modifications in the parameters change collective states. Any modification in expectations or, more generally, structural changes in economic and/or monetary conditions, may change expected returns and in turn the collective state. Unstable type-3 sectors are particularly sensitive to these changes in long-term growth, inflation and interest rates. Higher expectations in these sectors attract investment, which in turn increase expectations. This seemingly endless expected growth spirals until outlook flattens or deteriorates. A typical example would be the growth model of a company such as Amazon, whose ever-broadening product ranges has fuelled higher expected long-term returns and stock prices increase. Type-1 and -2 sectors attract capital through dividends and, although only partially and for high capital type-2 sectors, expected returns. Under higher expectations, these sectors are relatively less attractive than nearby type-3 sector. They may nonetheless survive in the long-term provided their short-term returns and dividends are high enough. This may be done by cutting costs or investment, at the expense of future growth. Moreover, advert signalling may emerge: an increase in dividends can be interpreted as faltering growth prospects. Conversely, any increase in long-term uncertainty impact expected returns and drive sector-3 capital towards other patterns. External shocks, inflation and monetary policy impact expectations, reduce long-term investment and either drive capital out of sectors 3 to sector-1 or -2, or favour other pattern-3 sectors.

Second, a deviation of capital from its collective state equilibrium value in one sector may itself initiate oscillations in the entire system. Actually, a temporary deviation from the given collective state implies an unstable redistribution of capital, expectations and returns. This generates interactions between sectors, reallocation of capital and global oscillations. These oscillations can dampen, or alternately drive the system towards a new collective state. There are thus potential transitions between collective states. This dynamics occurs at a slower, larger time-scale than that of market fluctuations. In the long-run, when transitions occur, both sectors' averages and patterns may have changed: patterns 2 may morph in, say, pattern 3 stable or unstable, sectors may simply disappear. Concretely, any significant modification in average capital in a sector could induce oscillations and initiate a transition.

Moreover, once endogenous expectations are introduced, they react to variations in capital, collective states of mixed 1-2-3-patterns are difficult to maintain. Highly reactive expectations favour patterns 3: expected returns magnify capital accumulation at the expense of other patterns. Mildly reactive expectations favour patterns 1 and 2: their oscillations, that are actually induced by uncertainties, dampen. Type-3 sectors on the contrary experience strong fluctuations in capital : attracting capital is less effective with fading expectations. The threshold in capital accumulation shifts upwards and least-profitable firms are ousted out of the sector. The recent evolution in performances between value and growth investment strategies exemplifies these shifts in investors' sentiment between expected growth and real returns. In periods of uncertainties, fluctuations affect capital accumulation in growth sectors, today's tech companies, and strengthen more dividend-driven investments. Note however that the most profitable and best capitalized firms, that remain above the threshold, maintain relatively high levels of capital. Here our versatile notion of firms proves convenient: any firm that accumulates enough capital to be able to buy back, in periods of volatility, its own

stocks is actually acting as an autonomous investor. When volatility is high, the most likely investors for the best capitalized firms are, first and foremost, the best capitalized companies themselves. They act so to speak as pools of closely held investors. In other words, provided firms have high enough capital, they can always cushion the impact of price fluctuations and adverse shocks through buybacks. Similarly, they also could choose to acquire companies in their sector or neighbouring sectors.

Fluctuations in financial expectations impose their pace to the real economy. Actually, expected returns are both exogenous and endogenous. Because they are exogenous, they may change quickly: expected returns, that theoretically should reflect long-term perspectives, actually rely on short-term sentiments: new information, changes in global economic outlook, adverse shocks regularly happen and modifies the long-term expectations. Capital moves from sectors to sectors quickly. But expected returns are also endogenous. Being expectations, they react, either highly or mildly, to changes within the system. When high levels of capital seek to maximize returns, we can suspect that expectations will react strongly to capital changes. The combination of expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital imply that large fluctuations of capital in the system. Creating or inflating expectations attracts capital, at times unduly. When this cannot be done, the sole remaining tool to reduce capital outflows is dividend policy, which may be done at the expense of labour force, capital expenditures and future growth.

12 Conclusion

We have studied the impact of financial capital on physical capital allocation and shown that collective states distribute sectors into several patterns of accumulation. All else equal, sectors with highest expected returns and capital may, through expectations, indefinitely attract capital at the expense of other sectors. This expansion is nevertheless unstable since adverse changes in expectations drive capital away.

At a macro timescale, the system can be globally described by oscillations between average capital and expected long-term returns, depending on the sectors' patterns. These oscillations, that can be either damp-ening or explosive, may change sectors patterns and explain switches from one collective state to another. Markets, supposedly the most efficient ressource allocation mechanism, add in a context of uncertainties impose their fluctuations to those of the real economy. This should render the role of Central Banks, or any kind of regulation, crucial to the good functioning of these markets.

Appendix 1 From large number of agents to field formalism

This appendix summarizes the most useful steps of the method developed in Gosselin, Lotz and Wambst (2017, 2020, 2021), to switch from the probabilistic description of the model to the field theoretic formalism and summarizes the translation of a generalisation of (14) involving different time variables. By convention and unless otherwise mentioned, the symbol $\int$ refers to all the variables involved.

A1.1 Probabilistic formalism

The probabilistic formalism for a system with N identical economic agents in interaction is based on the minimization functions described in the text. Classically, the dynamics derives through the optimization problem of these functions. The probabilitic formalism relies on the contrary on the fact, that, due to uncertainties, shocks... agents do not optimize fully these functions. Moreover, given the large number of agents, there may be some discrepency between agents minimization functions, and this fact may be translated in an uncertainty of behaviour around some average minimization, or objective funtion.

We thus assume that each agent chooses for his action a path randomly distributed around the optimal path. The agent's behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight.

In general, this weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget constraints and interactions among agents. These interactions may for instance arise through constraints, since income flows depend on other agents demand. The probabilistic description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given time span. They have the form of Euclidean path integrals.

In the context of the present paper, we have seen that the minimization functions for the system considered in this work have the form:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)^2 + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right) \right)$$

(164)

This minimization of this function will yield a dynamic equation for N agents in interaction described by a set of dynamic variables $\mathbf{A}_i(t)$ during a given timespan T.

The probabilistic description is straightforwardly obtained from (164). The probability associated to a configuration $(\mathbf{A}_i(t))_{\substack{i=1,...,N \\ 0 \leqslant t \leqslant T}}$ is directly given by:

$$\mathcal{N} \exp \left(-\frac{1}{\sigma^2} \int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)^2 \right. \right.$$

$$\left. \left. + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right) \right) \right)$$

(165)

where $\mathcal{N}$ is a normalization factor and σ^2 is a variance whose magnitude describes the amplitude of deviations around the optimal path.

As in the paper, the system is in general modelled by several equations, and thus, several minimization function. The overall system is thus described by several functions, and the minimization function of the

whole system is described by the set of functions:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)^2 + \sum_i \left(\sum_{j,k,l...} g^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right) \right)$$

(166)

where α runs over the set equations describing the system's dynamics. The associated weight is then:

$$\mathcal{N} \exp\left(- \left(\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)^2 \right. \right.$$

$$\left. \left. + \sum_{i,\alpha} \left(\sum_{j,k,l...} g^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right) \right) \right)$$

(167)

The appearance of the sum of minimization functions in the probabilitic weight (167) translates the hypothesis that the deviations with respect to the optimization of the functions (166) are assumed to be independent.

For a large number of agents, the system described by (167) involves a large number of variables $K_i(t)$, $P_i(t)$ and $X_i(t)$ that are difficult to handle. To overcome this difficulty, we consider the space H of complex functions defined on the space of a single agent's actions. The space H describes the collective behaviour of the system. Each function Ψ of H encodes a particular state of the system. We then associate to each function Ψ of H a statistical weight, i.e. a probability describing the state encoded in Ψ. This probability is written $\exp(-S(\Psi))$, where $S(\Psi)$ is a functional, i.e. the function of the function Ψ. The form of $S(\Psi)$ is derived directly from the form of (167) as detailed in the text. As seen from (167), this translation can in fact be directly obtained from the sum of "classical" minimization functions weighted by the factors $\frac{1}{\sigma_\alpha^2}$:

$$\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)^2 + \sum_{i,\alpha} \left(\sum_{j,k,l...} g^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right) \right)$$

This is this shortcut we used in the text.

A1.2 Interactions between agents at different times

A straightforward generalisation of (14) involve agents interactions at different times. The terms considered have the form:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} \int f\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) d\mathbf{t} \right)^2$$

$$+ \sum_i \sum_{j,k,l...} \int g\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) d\mathbf{t}$$

(168)

where $\mathbf{t}$ stands for $(t_i, t_j, t_k, t_l...)$ and $d\mathbf{t}$ stands for $dt_i dt_j dt_k dt_l...$

The translation is straightforward. We introduce a time variable θ on the field side and the fields write $|\Psi(\mathbf{A}, \theta)|^2$ and $\left|\hat{\Psi}\left(\hat{\mathbf{A}}, \hat{\theta}\right)\right|^2$. The second term in (168) becomes:

$$\sum_i \sum_j \sum_{j,k...} \int g\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) d\mathbf{t}$$

$$\rightarrow \int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'..., \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}\right) |\Psi(\mathbf{A}, \theta)|^2 \left|\Psi\left(\mathbf{A}', \theta'\right)\right|^2 \left|\Psi\left(\mathbf{A}'', \theta''\right)\right|^2 d\mathbf{A} d\mathbf{A}' d\mathbf{A}''$$

(169)

$$\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}, \hat{\theta}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}', \hat{\theta}'\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' d\boldsymbol{\theta} d\hat{\boldsymbol{\theta}}$$

where $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are the multivariables $(\theta, \theta', \theta''...)$ and $\left(\hat{\theta}, \hat{\theta}'...\right)$ respectively and $\mathbf{d\theta d\hat{\theta}}$ stands for $d\theta d\theta' d\theta''...$ and $d\hat{\theta} d\hat{\theta}'...$

Similarly, the first term in (168) translates as:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} \int f\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt} \right)^2 \tag{170}$$

$$\rightarrow \quad \int \Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} + \Lambda(\mathbf{A}, \theta) \right) \right) \Psi(\mathbf{A}, \theta) \, d\mathbf{A} d\theta \tag{171}$$

by:

$$\Lambda(\mathbf{A}, \theta) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'..., \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}\right) \left|\Psi\left(\mathbf{A}', \theta'\right)\right|^2 \left|\Psi\left(\mathbf{A}'', \theta''\right)\right|^2 d\mathbf{A}' d\mathbf{A}'' \tag{172}$$

$$\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}', \theta''\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \mathbf{d\bar{\theta} d\hat{\theta}}$$

with $\mathbf{d\bar{\theta}} = d\theta' d\theta''$.

Ultimately, as in the text, additional terms (22):

$$\Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_\theta \left(\frac{\sigma^2_\theta}{2} \nabla_\theta - 1 \right) \right) \Psi(\mathbf{A}, \theta) \tag{173}$$

$$+ \hat{\Psi}^\dagger\left(\hat{\mathbf{A}}, \theta\right) \left(-\nabla_\theta \left(\frac{\sigma^2_\theta}{2} \nabla_\theta - 1 \right) \right) \hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right) + \alpha \left|\Psi(\mathbf{A})\right|^2 + \alpha \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2$$

are included to the action functional to take into account for the time variable.

Appendix 2 expression of $\Psi(K, X)$ as function of financial variables

A2.1 Finding $\Psi(K, X)$: principle

In this paragraph, we give the principle of resolution for $\Psi(K, X)$ for an arbitrary function H. The full resolution for some particular cases is given below. Given a particular state $\hat{\Psi}$, we aim at minimizing the action functional $S_1 + S_2 + S_3 + S_4$. However, given our assumptions, the action functional $S_3 + S_4$ depends on $\Psi(K, X)$, through average quantities, and moreover, we have assumed that physical capital dynamics depends on financial accumulation. As a consequence, we can neglect, in first approximation, the impact of $\Psi(K, X)$ on $S_3 + S_4$ and consider rather the minimization of $S_1 + S_2$ which is given by:

$$S_1 + S_2 = -\int \Psi^\dagger(K, X) \left(\nabla_X \left(\frac{\sigma^2_X}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) - \tau \left(\int \left|\Psi(K', X)\right|^2 dK' \right) \right. \tag{174}$$

$$\left. + \nabla_K \left(\frac{\sigma^2_K}{2} \nabla_K + u\left(K, X, \Psi, \hat{\Psi}\right) \right) \right) \Psi(K, X) \, dK dX$$

with:

$$u\left(K, X, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon} \left(K - \int \frac{F_2(R(K, X)) G\left(X - \hat{X}\right)}{\int F_2(R(K, X)) G\left(X - \hat{X}\right) \|\Psi(K, X)\|^2} \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X} \right) \tag{175}$$

and:

$$\Gamma(K, X) = \int \frac{F_2(R(K, X)) G\left(X - \hat{X}\right)}{K \int F_2(R(K, X)) G\left(X - \hat{X}\right) \|\Psi(K, X)\|^2} \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right) - 1$$

This is done in two steps. First, we find $\Psi(X)$, the background field for X when K determined by X. We then find the corrections to the particular cases considered and compute $\Psi(K, X)$.

A2.1.1 Particular case: K **determined by** X

A simplification arises, assuming K adaptating to X. We assume that in first approximation K is a function of X, written K_X:

$$K = K_X = \int \frac{F_2\left(R\left(K_X, X\right)\right) G\left(X - \hat{X}\right)}{\int F_2\left(R\left(K'_{X'}, X'\right)\right) G\left(X' - \hat{X}\right) \|\Psi\left(X'\right)\|^2 dX'} \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right) \tag{176}$$

This means that for any sector X, the capital of all agents in this sector are equal. At the individual level, this corresponds to set $\frac{d}{dt} K_i\left(t\right) = 0$. The level of capital adapts faster than the motion in sector space and reaches quickly its equilibrium value. Incindently, (176) implies that $\Gamma\left(K, X\right) = 0$. Actually, using (176):

$$
\begin{aligned}
\Gamma\left(K, X\right) &= \int \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right)}{K \int F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right) \|\Psi\left(K, X\right)\|^2} \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right) - 1 \\
&= \int \frac{F_2\left(R\left(K, X\right)\right) G\left(X - \hat{X}\right) \hat{K} \left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\int F_2\left(R\left(K_X, X\right)\right) G\left(X - \hat{X}\right) \hat{K} \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 d\left(\hat{K}, \hat{X}\right)} - 1 \\
&= 0
\end{aligned}
$$

A2.1.1.1 Justification of approximation (176) Approximation (176) justifies in the following way. When F_2 is slowly varying with K, we perform the following change of variable in (174):

$$\Psi \rightarrow \Psi \exp\left(-\frac{\int u\left(K, X, \Psi, \hat{\Psi}\right) dK}{\sigma_K^2}\right) \simeq \Psi \exp\left(-\frac{1}{2\sigma_K^2}\varepsilon u^2\left(K, X, \Psi, \hat{\Psi}\right)\right)$$

$$\Psi^{\dagger} \rightarrow \Psi^{\dagger} \exp\left(\frac{1}{\sigma_K^2} \int u\left(K, X, \Psi, \hat{\Psi}\right) dK\right) \simeq \Psi^{\dagger} \exp\left(-\frac{1}{2\sigma_K^2}\varepsilon u^2\left(K, X, \Psi, \hat{\Psi}\right)\right)$$

and this replaces S_2 in (174) by:

$$-\int \Psi^{\dagger}\left(K, X\right) \left(\frac{\sigma_K^2}{2}\nabla_K^2 - \frac{u^2}{2\sigma_K^2}\left(K, X, \Psi, \hat{\Psi}\right) + \frac{1}{2}\nabla_K u\left(K, X, \Psi, \hat{\Psi}\right)\right) \Psi\left(K, X\right) dK dX \tag{177}$$

The change of variable modifies S_1 in (174). Actually, the derivative ∇_X acts on $\exp\left(-\frac{1}{2\sigma_K^2} u^2\left(K, X, \Psi, \hat{\Psi}\right)\right)$ and the term:

$$-\int \Psi^{\dagger}\left(K, X\right) \nabla_X \left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K, X\right) H\left(K\right)\right) \Psi\left(K, X\right) dK dX$$

becomes:

$$-\int \Psi^{\dagger}\left(X\right) \nabla_X \left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K, X\right) H\left(K\right)\right) \Psi\left(X\right) dK dX \tag{178}$$

$$+\varepsilon \int \Psi^{\dagger}\left(K, X\right) \left(\frac{\sigma_X^2}{2\sigma_K^2} u \nabla_X u\right) \nabla_X \Psi\left(K, X\right) dK dX + \varepsilon \int \Psi^{\dagger}\left(K, X\right) \left(\frac{\sigma_X^2}{2\sigma_K^2}\left(\left(\nabla_X u\right)^2 + u \nabla_X^2 u\right)\right) \Psi\left(K, X\right) dK dX$$

$$-\int \Psi^{\dagger}\left(K, X\right) \left(\varepsilon \frac{u \nabla_X u}{\sigma_K^2} \nabla_X R\left(K, X\right) H\left(K\right) + \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4}\left(u \nabla_X u\right)^2\right) \Psi\left(K, X\right) dK dX$$

Using that u is of order $\frac{1}{\varepsilon}$ (see(175)), the minimum of $S_1 + S_2$ is obtained when the potential:

$$\int \Psi^{\dagger}\left(K, X\right) \left(\frac{u^2}{2\sigma_K^2} - \frac{1}{2}\nabla_K u\right) \Psi\left(K, X\right) dK dX \tag{179}$$

$$+\varepsilon \int \Psi^{\dagger}\left(K, X\right) \left(\frac{\sigma_X^2}{2\sigma_K^2}\left(\left(\nabla_X u\right)^2 + u \nabla_X^2 u\right)\right) \Psi\left(K, X\right) dK dX$$

$$-\int \Psi^{\dagger}\left(K, X\right) \left(\varepsilon \frac{u \nabla_X u}{\sigma_K^2} \nabla_X R\left(K, X\right) H\left(K\right) + \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4}\left(u \nabla_X u\right)^2\right) \Psi\left(K, X\right) dK dX$$

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is nul. The dominant term in (179) for $\varepsilon << 1$ is:

$$\int \Psi^\dagger \left(K, X \right) \left(\frac{u^2}{2\sigma_K^2} - \varepsilon^2 \frac{\sigma_X^2}{2\sigma_K^4} \left(u \nabla_X u \right)^2 \right) \Psi \left(K, X \right) dK dX \tag{180}$$

For $\sigma_X^2 << \sigma_K^2$ it implies that the minimum for $S_1 + S_2$ is obtained for:

$$u \left(K, X, \Psi, \hat{\Psi} \right) \simeq 0$$

with solution (176).

A2.1.1.2 Rewriting the action $S_1 + S_2$ With our choice $G \left(X - \hat{X} \right) = \delta \left(X - \hat{X} \right)$ we find:

$$K_X = \frac{\int \hat{K} \left\| \hat{\Psi} \left(\hat{K}, X \right) \right\|^2 d\hat{K}}{\left\| \Psi \left(X \right) \right\|^2} \tag{181}$$

and $\Psi \left(K, X \right)$ becomes a function $\Psi \left(X \right)$:

$$\Psi \left(K, X \right) \to \Psi \left(X \right)$$

To find the action for $\Psi \left(X \right)$ we evaluate (179) using $u \left(K_X, X, \Psi, \hat{\Psi} \right) = 0$, and compute the first term in (180) for $\Psi \left(X \right) = \Psi \left(K_X, X \right) \delta \left(u \right)$ by replacing:

$$\delta \left(u \right) \to \frac{\exp \left(-\varepsilon u^2 \right)}{\sqrt{2\pi\varepsilon}}$$

We obtain:

$$- \int \Psi^\dagger \left(K, X \right) \left(\frac{\sigma_K^2}{2} \nabla_K^2 \right) \Psi \left(K, X \right) dK dX \quad = \quad \frac{\sigma_K^2}{2} \int \left| \Psi \left(X \right) \right|^2 dX \int \frac{\exp \left(-\varepsilon u^2 \right)}{\sqrt{2\pi\varepsilon}} \nabla_K^2 \frac{\exp \left(-\varepsilon u^2 \right)}{\sqrt{2\pi\varepsilon}} dK$$

$$\simeq \quad \frac{\sigma_K^2}{2\varepsilon} \int \left| \Psi \left(X \right) \right|^2 dX$$

and the action S_1 restricted to the variable X is given by:

$$S_1 \quad = \quad \int \Psi^\dagger \left(X \right) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \left(\nabla_X R \left(X \right) H \left(K_X \right) \right) \right) + \tau \left| \Psi \left(X \right) \right|^2 \right) \Psi \left(X \right)$$

$$+ \int \Psi^\dagger \left(K, X \right) \left(\frac{\sigma_X^2}{4\sigma_K^2} \left(\nabla_X u \left(K_X, X, \Psi, \hat{\Psi} \right) \right)^2 \right) \Psi \left(K, X \right) dK dX$$

$$+ \int \left(\frac{\sigma_K^2}{2\varepsilon} - \frac{1}{2} \nabla_K u \left(K_X, X, \Psi, \hat{\Psi} \right) \right) \left| \Psi \left(X \right) \right|^2 dX$$

In our order of appromation $\nabla_K u \left(K_X, X, \Psi, \hat{\Psi} \right) \simeq \varepsilon$. Ultimately, for $\sigma_X^2 << \sigma_K^2$, action S_1 reduces to:

$$S_1 = \int \Psi^\dagger \left(X \right) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \left(\nabla_X R \left(X \right) H \left(K_X \right) \right) \right) + \tau \left| \Psi \left(X \right) \right|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi \left(X \right) \tag{182}$$

and we look for $\Psi \left(X \right)$ minimizing (182).

A2.1.1.3 Minimization of (182) To minimize (182), we assume for the sake of simplicity, that for $i \neq j$:

$$\left| \nabla_{X_i} \nabla_{X_j} R(X) \right| << \left| \nabla_{X_i}^2 R(X) \right|$$

which is the case for example if $R(X)$ is a function with separated variables : $R(X) = \sum R_i(X_i)$. This can be also realized if locally, one chooses the variables X_i to diagonalize $\nabla_{X_i} \nabla_{X_j} R(X)$ at some points in the sector space.

We then perform the change of variables:

$$\exp\left(\int^X \frac{\nabla_X R(X)}{\sigma_X^2 \|\nabla_X R(X)\|} H(K_X) \right) \Psi(X) \to \Psi(X)$$

and:

$$\exp\left(-\int^X \nabla_X R(X) H(K_X) \right) \Psi^\dagger(X) \to \Psi^\dagger(X)$$

so that (182) becomes:

$$\int \Psi^\dagger(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R(X) H(K_X) \right)^2 + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X) \tag{183}$$

which is of second order in derivatives with a potential:

$$\tau \|\Psi(X)\|^4 + \frac{1}{2\sigma_X^2} \int \left(\nabla_X R(X) H(K_X) \right)^2 \|\Psi(X)\|^2$$

We assume the number of agents fixed equal to N. We have to minimize (183) with the contraint $\|\Psi(X)\|^2 \geqslant 0$ and $\int \|\Psi(X)\|^2 = N$. We thus replace (183) by:

$$\int \Psi^\dagger(X) \left(-\frac{\sigma_X^2 \nabla_X^2}{2} + \frac{\left(\nabla_X R(X) H(K_X) \right)^2}{2\sigma_X^2} + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau |\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon} \right) \Psi(X)$$

$$+ D\left(\|\Psi\|^2 \right) \left(\int \|\Psi(X)\|^2 - N \right) + \int \mu(X) \|\Psi(X)\|^2 \tag{184}$$

we have written $D\left(\|\Psi\|^2 \right)$ the Lagrange multiplier for $\int \|\Psi(X)\|^2$, to keep track of its dependency multiplier in $\|\Psi\|^2$. By a redefinition $D\left(\|\Psi\|^2 \right) - \frac{\sigma_K^2 - 1}{2\varepsilon} \to D\left(\|\Psi\|^2 \right)$, $\frac{D\left(\|\Psi\|^2 \right)}{D\left(\|\Psi\|^2 \right) - \frac{\sigma_K^2}{2\varepsilon}} N \to N$ we can write (184) as:

$$\int \Psi^\dagger(X) \left(-\frac{\sigma_X^2}{2} \nabla_X^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R(X) H(K_X) \right)^2 + \frac{H(K_X) \nabla_X^2 R(K_X, X)}{2} + \tau |\Psi(X)|^2 \right) \Psi(X) \tag{185}$$

$$+ D\left(\|\Psi\|^2 \right) \left(\int \|\Psi(X)\|^2 - N \right) + \int \mu(X) \|\Psi(X)\|^2$$

Introducing the change of variable for $\nabla_X R(X)$ for the sake of simplicity:

$$\left(\nabla_X R(X) \right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \to \left(\nabla_X R(X) \right)^2 \tag{186}$$

the minimization of the potential yields, for $\sigma_X^2 << 1$:

$$iD\left(\|\Psi\|^2\right) + \mu(X) \tag{187}$$

$$= 2\tau\|\Psi(X)\|^2 - \frac{H'\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}{2\sigma_X^2 H\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}$$

$$\times \left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^4}\|\Psi(X)\|^2$$

$$+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2$$

Moreover, multiplying (187) by $\|\Psi(X)\|^2$ and integrating yields:

$$D\left(\|\Psi\|^2\right) N = 2\tau \int |\Psi(X)|^4 \tag{188}$$

$$- \int \frac{H'\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}{2\sigma_X^2 H\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}\left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}$$

$$+\frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \|\Psi(X)\|^2$$

$$\simeq 2\tau \int |\Psi(X)|^4 + \frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \|\Psi(X)\|^2$$

Note that in first approximation, for $H' << 1$, (187) and (188) become:

$$D\left(\|\Psi\|^2\right) + \mu(X) = 2\tau\|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2}(\nabla_X R(X))^2 H^2\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right) \tag{189}$$

and:

$$ND\left(\|\Psi\|^2\right) = 2\tau\int|\Psi(X)|^4 + \frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X) H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2 \|\Psi(X)\|^2 \tag{190}$$

A2.1.1.4 Resolution of (189) and (190) Two cases arise in the resolution:

Case 1: $\|\Psi(X)\|^2 > 0$ For $\|\Psi(X)\|^2 > 0$, (187) writes:

$$D\left(\|\Psi\|^2\right) = 2\tau\|\Psi(X)\|^2 + \frac{1}{2\sigma_X^2}(\nabla_X R(X))^2 H^2\left(\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right) \tag{191a}$$

with:

$$\hat{K}_X = \int \hat{K} \left\| \hat{\Psi} \left(\hat{K}, X \right) \right\|^2 d\hat{K} = K_X \left\| \Psi \left(X \right) \right\|^2 \tag{192}$$

Note that restoring the initial variable:

$$\left(\nabla_X R \left(X \right) \right)^2 \to \left(\nabla_X R \left(X \right) \right)^2 + \sigma_X^2 \frac{\nabla_X^2 R \left(K_X, X \right)}{H \left(K_X \right)} \tag{193}$$

yields (59) in the text.

Given the setup, we can assume that

$$H^2 \left(\frac{\hat{K}_X}{\left\| \Psi \left(X \right) \right\|^2} \right) \left(1 - \frac{H' \left(\hat{K}_X \right)}{H \left(\hat{K}_X \right)} \frac{\hat{K}_X}{\left\| \Psi \left(X \right) \right\|^2} \right)$$

is a decreasing function of $\left\| \Psi \left(X \right) \right\|^2$. Assume a minimum $\Psi_0 \left(X \right)$ for the right hand side of (191a). It leads to a condition for $D \left(\left\| \Psi \right\|^2 \right)$:

$$D \left(\left\| \Psi \right\|^2 \right) > 2\tau \left\| \Psi_0 \left(X \right) \right\|^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R \left(X \right) \right)^2 H^2 \left(\frac{\hat{K}_X}{\left\| \Psi_0 \left(X \right) \right\|^2} \right) \left(1 - \frac{H' \left(\hat{K}_X \right)}{H \left(\hat{K}_X \right)} \frac{\hat{K}_X}{\left\| \Psi_0 \left(X \right) \right\|^2} \right) \tag{194}$$

and the solution of (191a) writes:

$$\left\| \Psi \left(X, \left(\nabla_X R \left(X \right) \right)^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \right\|^2 \tag{195}$$

where $\hat{K}_{X,0}$ is a constant representing some average to normalize $\frac{\hat{K}_X}{\hat{K}_{X,0}}$ as a dimensionless number.

Case 2 $\left\| \Psi \left(X \right) \right\|^2 = 0$ On the other hand, if:

$$D \left(\left\| \Psi \right\|^2 \right) < 2\tau \left\| \Psi_0 \left(X \right) \right\|^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R \left(X \right) \right)^2 H^2 \left(\frac{\hat{K}_X}{\left\| \Psi_0 \left(X \right) \right\|^2} \right) \left(1 - \frac{H' \left(\hat{K}_X \right)}{H \left(\hat{K}_X \right)} \frac{\hat{K}_X}{\left\| \Psi_0 \left(X \right) \right\|^2} \right) \tag{196}$$

the solution of (191a) is $\left\| \Psi \left(X \right) \right\|^2 = 0$

Gathering both cases   The value of $\left\| \Psi \right\|^2$ thus depends on the conditions (194) and (196). To compute the value of $D \left(\left\| \Psi \right\|^2 \right)$ we integrate (191a) over V/V_0 with V_0 locus where $\left\| \Psi \left(X \right) \right\|^2 = 0$. V_0 will be then defined by (196) once $D \left(\left\| \Psi \right\|^2 \right)$ found. For H slowly varying, we can replace $\frac{\hat{K}_X}{\left\| \Psi \left(X \right) \right\|^2}$ by:

$$\frac{\int \hat{K} \left\| \hat{\Psi} \left(\hat{K}, X \right) \right\|^2 d\hat{K} dX}{\int \left\| \Psi \left(X \right) \right\|^2 dX} = \frac{\int \hat{K} \left\| \hat{\Psi} \left(\hat{K}, X \right) \right\|^2 d\hat{K} dX}{N}$$

so that the integration of (196) over X yields:

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$$D\left(\|\Psi\|^2\right)(V - V_0) \simeq 2\tau N + \frac{1}{2\sigma_X^2}\int (\nabla_X R(X))^2 H^2 \left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}dX}{N}\right)$$

$$\times \left(1 - \frac{H'\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}dX}{N}\right)}{H\left(\frac{\int \hat{K}\|\hat{\Psi}(\hat{K},X)\|^2 d\hat{K}dX}{N}\right)}\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}dX}{N}\right)$$

$$= 2\tau N + \frac{1}{2\sigma_X^2}(\nabla_X R(X))^2 H^2\left(\frac{\langle\hat{K}\rangle}{N}\right)\left(1 - \frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\langle\hat{K}\rangle}{N}\right)$$

As a consequence:

$$D\left(\|\Psi\|^2\right) \simeq 2\tau\frac{N}{V - V_0} + \frac{1}{2\sigma_X^2}\left\langle(\nabla_X R(X))^2\right\rangle_{V/V_0} H^2\left(\frac{\langle\hat{K}\rangle}{N}\right)\left(1 - \frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\langle\hat{K}\rangle}{N}\right)$$

and V_0 is defined by (196):

$$2\tau\frac{N}{V - V_0} + \frac{1}{2\sigma_X^2}\left\langle(\nabla_X R(X))^2\right\rangle_{V/V_0} H^2\left(\frac{\langle\hat{K}\rangle}{N}\right)\left(1 - \frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\langle\hat{K}\rangle}{N}\right) \tag{197}$$

$$< 2\tau\|\Psi_0(X)\|^2 + \frac{1}{2\sigma_X^2}(\nabla_X R(X))^2 H^2\left(\frac{\hat{K}_X}{\|\Psi_0(X)\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\|\Psi_0(X)\|^2}\right)$$

On V/V_0, $\|\Psi\|^2$ is given by (195) and on V_0, $\|\Psi\|^2 = 0$.

Below, we give explicitely the form of $\Psi(X)$ form two different form of the function H.

A2.1.2 Introducing the K dependency

A2.1.2.1 First order condition To go beyond approximation (176) and solve for the field $\Psi(K, X)$ that minimizes (174), we come back to the full system for K and X:

$$\int \Psi^\dagger(K, X)\left(\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \left(\frac{\nabla_X R(K, X)}{\|\nabla_X R(K, X)\|}\right)H(K) + \tau|\Psi(K, X)|^2\right)\right) \tag{198}$$

$$-\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K, X, \Psi, \hat{\Psi}\right)\right) - \frac{1}{2}\nabla_K u\left(K, X, \Psi, \hat{\Psi}\right)\right)\Psi(K, X)$$

with $u\left(K, X, \Psi, \hat{\Psi}\right)$ given by (175). We then look for a minimum of (198) of the form:

$$\Psi(K, X) = \Psi(X)\Psi_1(K - K_X) \tag{199}$$

with K_X given in (181):

$$K_X = \frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{200}$$

and Ψ_1 peaked around 0 and of norm 1. When $H(K)$ is slowly varying around K_X, the minimization of (198) for $\Psi_1(K - K_X)$ writes:

$$\nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + u\left(K, X, \Psi, \hat{\Psi}\right) + \frac{1}{2}\nabla_K u\left(K, X, \Psi, \hat{\Psi}\right) \right) \Psi_1(K - K_X) = 0 \qquad (201)$$

Then, using that, in first approximation:

$$\int F_2\left(R\left(K', X\right)\right) \|\Psi(K', X)\|^2 \, dK' \simeq F_2\left(R\left(K_X, X\right)\right) \|\Psi(X)\|^2$$

Equation (201) becomes:

$$\nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + K - \frac{F_2\left(R\left(K, X\right)\right) K_X}{F_2\left(R\left(K_X, X\right)\right)} \right) \Psi_1(K - K_X) = 0 \qquad (202)$$

A2.1.2.2 Solving (202) To solve the first order condition (202) we perform the change of variable:

$$\Psi_1(K - K_X) \rightarrow \exp\left(\frac{1}{\sigma_K^2} \int \left[K - \frac{F_2\left(R\left(K, X\right)\right) K_X}{F_2\left(R\left(K_X, X\right)\right)} \right] dK \right) \Psi_1(K - K_X)$$

and (202) is transformed into

$$-\frac{\sigma_K^2}{2} \nabla_K^2 \Psi_1(K - K_X) + \frac{1}{2\sigma_K^2} \left(K - \frac{F_2\left(R\left(K, X\right)\right) K_X}{F_2\left(R\left(K_X, X\right)\right)} \right)^2 \Psi_1(K - K_X) = 0 \qquad (203)$$

This equation can be solved by implementing the constraint:

$$\int \|\Psi_1(K - K_X)\|^2 = 1$$

and we find:

$$\Psi_1(K - K_X) \simeq \mathcal{N} \exp\left(-\frac{1}{\sigma_K^2} \left(K - \frac{F_2\left(R\left(K, X\right)\right) K_X}{F_2\left(R\left(K_X, X\right)\right)} \right)^2 \right)$$

$$\simeq \mathcal{N} \exp\left(-\frac{1}{\sigma_K^2} \left(K - K_X - (K - K_X) \frac{\partial_K R\left(K_X, X\right) F_2'\left(R\left(K_X, X\right)\right)}{F_2\left(R\left(K_X, X\right)\right)} K_X \right)^2 \right)$$

$$= \mathcal{N} \exp\left(-\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R\left(K_X, X\right) F_2'\left(R\left(K, X\right)\right)}{F_2\left(R\left(K_X, X\right)\right)} K_X \right)^2 (K - K_X)^2 \right)$$

with the normalization factor $\mathcal{N}$ given by:

$$\mathcal{N} = \sqrt{\frac{c}{\sigma_K^2 \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K, X))}{F_2(R(K_X, X))} K_X \right)^2}}$$

A2.1.2.3 Expression for the density of firms $\|\Psi(K, X)\|^2$ Having found Ψ_1, and using (195) and (199) we obtain the expression for $\|\Psi(K, X)\|^2$:

$$\|\Psi(K, X)\|^2 = \mathcal{N} \|\Psi\|^2 \left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \qquad (204)$$

$$\times \exp\left(-\frac{1}{\sigma_K^2} \left(K - \frac{F_2\left(R\left(K, X\right)\right)}{F_2\left(R\left(K_X, X\right)\right) \|\Psi(X)\|^2} \int \hat{K} \left\| \hat{\Psi}\left(\hat{K}, X\right) \right\|^2 d\hat{K} \right)^2 \right)$$

$$= \|\Psi\|^2 \left(X, (\nabla_X R(X))^2, \frac{\hat{K}_X}{\hat{K}_{X,0}} \right) \frac{c \exp\left(-\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K, X))}{F_2(R(K_X, X))} K_X \right)^2 (K - K_X)^2 \right)}{\frac{1}{\sigma_K^2} \left(1 - \frac{\partial_K R(K_X, X) F_2'(R(K, X))}{F_2(R(K_X, X))} K_X \right)^2}$$

for $X \in V/V_0$ and $\|\Psi\left(K,X\right)\|^2 = 0$ otherwise.

As stated in the text, note that the form of the exponential in (204) implies that:

$$\int K \left\|\Psi\left(K,X\right)\right\|^2 d\hat{K} = \int \hat{K} \left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}$$

A2.2 Examples

A2.2.1 Example 1

We compute $\Psi\left(K,X\right)$ for the specific function:

$$H\left(y\right) = \left(\frac{y}{1+y}\right)^{\varsigma}, \ H'\left(y\right) = \varsigma \frac{\left(\frac{y}{y+1}\right)^{\varsigma}}{y\left(y+1\right)}$$

We use the simplified equations (189) and (190) that yield:

$$D\left(\|\Psi\|^2\right) + \mu\left(X\right) = \tau\|\Psi\left(X\right)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \left(\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi\left(X\right)\|^2}\right)^{\varsigma}\right)^2 \left(1 - \varsigma \frac{1}{\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi\left(X\right)\|^2} + \left\langle\hat{K}\right\rangle\right)}\right)}{\left(\left\langle\hat{K}\right\rangle + \frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi\left(X\right)\|^2}\right)^{2\varsigma}}$$

or equivalently:

$$\begin{aligned} D\left(\|\Psi\|^2\right) + \mu\left(X\right) &= \tau\|\Psi\left(X\right)\|^2 \\ &+ \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \left(\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right)^{2\varsigma}}{\left(\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2 + \int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right)^{2\varsigma+1}} \\ &\times \left(\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K} + \left(1-\varsigma\right)\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2\right) \end{aligned}$$

For $\varsigma \simeq \frac{1}{2}$, this reduces to:

$$D\left(\|\Psi\|^2\right) + \mu\left(X\right) = \tau\|\Psi\left(X\right)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \hat{K}_X\left(\hat{K}_X + \frac{1}{2}\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2\right)}{\left(\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2 + \hat{K}_X\right)^2}$$

and for $\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2 << \hat{K}_X$ this becomes:

$$D\left(\|\Psi\|^2\right) + \mu\left(X\right) \simeq \tau\|\Psi\left(X\right)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \hat{K}_X}{\left(\left\langle\hat{K}\right\rangle\|\Psi\left(X\right)\|^2 + \hat{K}_X\right)} \tag{205}$$

Two cases arise.

When $\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 << \tau$:

$$\begin{aligned} \|\Psi\left(X\right)\|^2 &= \frac{\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\right) + \sqrt{\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\right)^2 - 4\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\left(\frac{\left(\nabla_X R\left(X\right)\right)^2}{\sigma_X^2} - D\left(\|\Psi\|^2\right)\right)}}{2\tau} \\ &= \frac{4\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\left(\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right)\right)}{2\tau\left(\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\right) - \sqrt{\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\right)^2 - 4\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\left(\frac{\left(\nabla_X R\left(X\right)\right)^2}{\sigma_X^2} - D\left(\|\Psi\|^2\right)\right)}\right)} \end{aligned} \tag{206}$$

This is positive on the set:

$$\left\{\left(D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right) > 0\right\} \cup \left\{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) < 0\right\} \tag{207}$$

To detail these two conditions, we write (205) for $\|\Psi\left(X\right)\|^2 > 0$:

$$D\left(\|\Psi\|^2\right) \simeq \tau \|\Psi\left(X\right)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}}{\left(\|\Psi\left(X\right)\|^2 + \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right)}$$

which is equivalent to:

$$\frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right)}{\left(\|\Psi\left(X\right)\|^2 + \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right)} \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle} = \frac{-\tau \|\Psi\left(X\right)\|^2 + D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}}{\left(\|\Psi\left(X\right)\|^2 + \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right)} \|\Psi\left(X\right)\|^2$$

Then, we have the implication:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) > 0 \Rightarrow D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle} > 0 \tag{208}$$

This implies that (207) is always satisfied, and formula (206) is valid for all X.

The second case arises when $\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 << \tau$. In this case, the solution is:

$$\|\Psi\left(X\right)\|^2 = \frac{\left(D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right) - \sqrt{\left(D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\right)^2 - 4\tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle}\left(\frac{\left(\nabla_X R\left(X\right)\right)^2}{\sigma_X^2} - D\left(\|\Psi\|^2\right)\right)}}{2\tau}$$

This solution is valid, i.e. $\|\Psi\left(X\right)\|^2 > 0$, under the conditions:

$$\left\{D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle} > 0\right\} \cap \left\{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) > 0\right\} \tag{209}$$

and $\|\Psi\|^2 = 0$ for:

$$\left\{D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle} < 0\right\} \cup \left\{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) < 0\right\}$$

To detail these two conditions, we use the implication (208) that is equivalent to:

$$D\left(\|\Psi\|^2\right) - \tau \frac{\hat{K}_X}{\left\langle \hat{K} \right\rangle} < 0 \Rightarrow \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) < 0$$

As a consequence, $\|\Psi\left(X\right)\|^2 = 0$ only if:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 - D\left(\|\Psi\|^2\right) < 0 \tag{210}$$

We find $D\left(\|\Psi\|^2\right)$ by integration of:

$$D\left(\|\Psi\|^2\right) + \mu\left(X\right) \simeq \tau\|\Psi\left(X\right)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \hat{K}_X}{\left(\left\langle \hat{K}\right\rangle \|\Psi\left(X\right)\|^2 + \hat{K}_X\right)} \tag{211}$$

and this leads to:

$$
\begin{aligned}
\int_{V/V_0} D\left(\|\Psi\|^2\right) &\simeq \tau N + \int_{V/V_0} \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \frac{\hat{K}_X}{\left\langle \hat{K}\right\rangle}}{\left(\|\Psi\left(X\right)\|^2 + \frac{\hat{K}_X}{\left\langle \hat{K}\right\rangle}\right)} \\
&\simeq \tau N + \frac{1}{2}\int_{V/V_0} \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 = \tau N + \frac{1}{2}\left(V - V_0\right)\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0}
\end{aligned}
$$

we thus have:

$$D\left(\|\Psi\|^2\right) \simeq \frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0} \tag{212}$$

and V_0 is defined using (210). It is the set of points X such that :

$$\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0} - \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 > 0 \tag{213}$$

Similarly, the set V/V_0 is defined by:

$$\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0} - \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 < 0 \tag{214}$$

To each function $R\left(X\right)$ and any $d > 0$, we associate two functions that depend on the form of $\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2$ over the whole space. First, $v\left(V - V_0\right)$ is a decreasing function of $V - V_0$, defined by:

$$V\left(\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 > v\left(V - V_0\right)\right) = V - V_0 \tag{215}$$

Second, for every $d \geqslant 0$, the function $h\left(d\right)$ is given by:

$$h\left(d\right) = \frac{1}{\int_{\nabla_X R(X)>d} dX}\int_{\nabla_X R(X)>d} \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 dX \tag{216}$$

This is an increasing function of d.

Thus, we can rewrite (214) as:

$$\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0} = v\left(V - V_0\right) \tag{217}$$

and moreover, by integration of (214) over V/V_0:

$$\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0} = h\left(\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 \right\rangle_{V/V_0}\right) \tag{218}$$

Equations (217) and (218) combine as:

$$2\left(v\left(V - V_0\right) - \frac{\tau N}{V - V_0}\right) = h\left(v\left(V - V_0\right)\right) \tag{219}$$

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which is an equation depending on the form of $R(X)$. If it has a solution, the set on which $\|\Psi(X)\|^2 = 0$ is defined by:

$$\frac{1}{\sigma_X^2}(\nabla_X R(X))^2 < v(V - V_0)$$

and $D\left(\|\Psi\|^2\right)$ is given by

$$D\left(\|\Psi\|^2\right) \simeq v(V - V_0)$$

Once the solution of (219) is known, the constant $D\left(\|\Psi\|^2\right)$ is given by (212) and:

$$\|\Psi(X)\|^2 = \frac{2\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{1}{\sigma_X^2}(\nabla_X R(X))^2 - D\left(\|\Psi\|^2\right)\right)}{D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\langle\hat{K}\rangle} + \sqrt{\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)^2 - 4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{1}{\sigma_X^2}(\nabla_X R(X))^2 - D\left(\|\Psi\|^2\right)\right)}}$$

(220)

for $X \in V/V_0$.

A2.2.2 Example 2

We choose $H(y) = y$ and equations (189) and (190) yield:

$$D\left(\|\Psi\|^2\right) \simeq \tau\|\Psi(X)\|^2 + \frac{1}{\sigma_X^2}(\nabla_X R(X))^2 \frac{\hat{K}_X}{\|\Psi(X)\|^2}$$

If:

$$D\left(\|\Psi\|^2\right) > 2\sqrt{\tau\frac{1}{\sigma_X^2}(\nabla_X R(X))^2 \hat{K}_X}$$

(221)

then:

$$\|\Psi(X)\|^2 = \frac{1}{2\tau}\left(D\left(\|\Psi\|^2\right) - \sqrt{\left(D\left(\|\Psi\|^2\right)\right)^2 - 4\hat{K}_X\frac{1}{\sigma_X^2}(\nabla_X R(X))^2\tau}\right) > 0$$

To solve (212) and to find V_0, we compute $D\left(\|\Psi\|^2\right)$ by integrating (211) and (212) is still valid:

$$D\left(\|\Psi\|^2\right) \simeq \frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle\frac{1}{\sigma_X^2}(\nabla_X R(X))^2\right\rangle_{V/V_0}$$

(222)

We proceed as in the previous paragraph to find $D\left(\|\Psi\|^2\right)$ and V_0. Using (222), (221) becomes:

$$\frac{1}{4\tau\hat{K}_X}\left(\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle\frac{1}{\sigma_X^2}(\nabla_X R(X))^2\right\rangle_{V/V_0}\right)^2 > \frac{1}{\sigma_X^2}(\nabla_X R(X))^2$$

(223)

Definitions (215) and (216) allow to rewrite (222) and (223):

$$\frac{1}{4\tau\hat{K}_X}\left(\frac{\tau N}{V - V_0} + \frac{1}{2}\left\langle\frac{1}{\sigma_X^2}(\nabla_X R(X))^2\right\rangle_{V/V_0}\right)^2 = v(V - V_0)$$

$$\left\langle\frac{1}{\sigma_X^2}(\nabla_X R(X))^2\right\rangle_{V/V_0} = h(v(V - V_0))$$

that reduce to an equation for $V - V_0$:

$$2\left(2\sqrt{\tau v(V - V_0)\hat{K}_X} - \frac{\tau N}{V - V_0}\right) = h(v(V - V_0))$$

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If it has a solution, the set on which $\|\Psi(X)\|^2 = 0$ is defined by:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 < v\left(V - V_0\right)$$

and $D\left(\|\Psi\|^2\right)$ is given by

$$D\left(\|\Psi\|^2\right) \simeq 2\sqrt{\tau v\left(V - V_0\right)\hat{K}_X}$$

Appendix 3. Computation of the background field $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ and average capital $\hat{K}_X$

A3.1 System for $\hat{\Psi}\left(\hat{K},\hat{X}\right)$

A3.1.1 Replacing quantities depending on (K,X)

Having found $\Psi\left(K,X\right)$, we can rewrite an action functional for $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. To do so, we first replace the quantities depending on $\Psi\left(K,X\right)$ in the action (42). Given the form of this function we can use the approximation $K \simeq K_X$: at the collective level, the relevant quantity, from the point of view of investors are the sectors.

Using that:

$$\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)} \simeq \frac{R\left(K,X\right)}{\int R\left(K'_{X'},X'\right)\|\Psi\left(X'\right)\|^2 dX'}$$

we first start by rewriting F_1 and we have:

$$F_1\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\|\Psi\left(K',X'\right)\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right) \simeq F_1\left(\frac{R\left(K_X,X\right)}{\int R\left(K'_{X'},X'\right)\|\Psi\left(X'\right)\|^2 dX'},\Gamma\left(K,X\right)\right)$$

As explained in appendix 1, when $K \simeq K_X$, we also have:

$$\Gamma\left(K,X\right) = \int \frac{F_2\left(R\left(K,X\right)\right)}{K_X F_2\left(R\left(K_X,X\right)\right)\|\Psi\left(K_X,X\right)\|}\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K} - 1 = 0$$

Then, we rewrite the expression involving F_2 in (42):

$$\frac{F_2\left(R\left(K,\hat{X}\right)\right)}{\int F_2\left(R\left(K',\hat{X}\right)\right)\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\left\|\Psi\left(K,\hat{X}\right)\right\|^2 \simeq \frac{F_2\left(R\left(K,\hat{X}\right)\right)}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2}\left\|\Psi\left(K,\hat{X}\right)\right\|^2$$

$$= \frac{F_2\left(R\left(K,\hat{X}\right)\right)\left\|\Psi_0\left(K - K_{\hat{X}}\right)\right\|^2}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}$$

and the $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ part of the action functional (42) writes:

$$S_3 + S_4 = -\int \hat{\Psi}^\dagger\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(K,X,\Psi,\hat{\Psi}\right)\right)\right) \tag{224}$$

$$+\nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)$$

where:

$$
\begin{aligned}
f\left(\hat{X}, \Psi, \hat{\Psi}\right) &= \frac{1}{\varepsilon} \int \left(\nabla_K R\left(K, X\right) - \gamma \frac{\int K' \left\|\Psi\left(K', X\right)\right\|^2}{K} + F_1 \left(\frac{R\left(K, X\right)}{\int R\left(K', X'\right) \left\|\Psi\left(K', X'\right)\right\|^2 d\left(K', X'\right)} \right) \right) \\
&\quad \times \frac{F_2\left(R\left(K, \hat{X}\right)\right) \left\|\Psi_0\left(K - K_{\hat{X}}\right)\right\|^2}{F_2\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)} dK
\end{aligned}
\tag{225}
$$

$$
\begin{aligned}
g\left(\hat{X}, \Psi, \hat{\Psi}\right) &= \int \left(\frac{\nabla_{\hat{X}} F_0\left(R\left(K, \hat{X}\right)\right)}{\left\|\nabla_{\hat{X}} R\left(K, \hat{X}\right)\right\|} + \nu \nabla_{\hat{X}} F_1 \left(\frac{R\left(K, \hat{X}\right)}{\int R\left(K', X'\right) \left\|\Psi\left(K', X'\right)\right\|^2 d\left(K', X'\right)}, \Gamma\left(K, X\right) \right) \right) \\
&\quad \times \frac{\left\|\Psi\left(K, \hat{X}\right)\right\|^2 dK}{\int \left\|\Psi\left(K', \hat{X}\right)\right\|^2 dK'}
\end{aligned}
\tag{226}
$$

An other simplification arises for the function $F_2\left(R\left(K, \hat{X}\right)\right)$. Actually:

$$
\begin{aligned}
&\frac{F_2\left(R\left(K, \hat{X}\right)\right)}{\int F_2\left(R\left(K', \hat{X}\right)\right) \left\|\Psi\left(K', \hat{X}\right)\right\|^2 dK'} \left\|\Psi\left(K, \hat{X}\right)\right\|^2 \\
&\simeq \frac{F_2\left(R\left(K, \hat{X}\right)\right)}{\int F_2\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \left\|\Psi\left(\hat{X}\right)\right\|^2} \left\|\Psi\left(K, \hat{X}\right)\right\|^2 \\
&\simeq \frac{F_2\left(R\left(K, \hat{X}\right)\right)}{F_2\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)} \left\|\Psi\left(K - K_{\hat{X}}\right)\right\|^2
\end{aligned}
$$

and by integration in (225) and (226), we have:

$$
f\left(\hat{X}, \Psi, \hat{\Psi}\right) = \frac{1}{\varepsilon} \left(r\left(K_{\hat{X}}, \hat{X}\right) - \gamma \left\|\Psi\left(\hat{X}\right)\right\|^2 + F_1\left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right) \left\|\Psi\left(X'\right)\right\|^2 dX'} \right) \right)
\tag{227}
$$

$$
g\left(\hat{X}, \Psi, \hat{\Psi}\right) = \frac{\nabla_{\hat{X}} F_0\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}{\left\|\nabla_{\hat{X}} R\left(K_{\hat{X}}, \hat{X}\right)\right\|} + \nu \nabla_{\hat{X}} F_1\left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right) \left\|\Psi\left(X'\right)\right\|^2 dX'} \right)
$$

In the sequel, for the sake of simplicity, we will write $f\left(\hat{X}\right)$ and $g\left(\hat{X}\right)$ for $f\left(\hat{X}, K_{\hat{X}}\right)$ and $g\left(\hat{X}, K_{\hat{X}}\right)$ respectively. We then perform the following change of variable in (224):

$$
\begin{aligned}
\hat{\Psi} &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^2} \int g\left(\hat{X}\right) d\hat{X} \right) \hat{\Psi} \\
\hat{\Psi}^\dagger &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^2} \int g\left(\hat{X}\right) d\hat{X} \right) \hat{\Psi}^\dagger
\end{aligned}
$$

so that (224) becomes:

$$
\begin{aligned}
S_3 + S_4 = &-\int \hat{\Psi}^\dagger \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g\left(\hat{X}, K_{\hat{X}}\right) \right)^2 - \frac{1}{2} \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \hat{\Psi} \\
&-\int \hat{\Psi}^\dagger \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f\left(\hat{X}, K_{\hat{X}}\right) \right) \right) \hat{\Psi}
\end{aligned}
\tag{228}
$$

This action functional for $\hat{\Psi}$ will be minimized in the next paragraph. Note that we should also include to (228), the action functional $S_1 + S_2$ evaluated at the background field Ψ, since this ones depends on $\hat{\Psi}$. However, we have seen that at the background field Ψ, for $K \simeq K_X$, $u\left(K, X, \Psi, \hat{\Psi}\right) \simeq 0$ and the action functional $S_1 + S_2$ defined in (174) reduces to:

$$S_1 + S_2 \simeq \int \Psi^\dagger\left(X\right)\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \left(\nabla_X R\left(X\right) H\left(K_X\right)\right)\right) + \tau\left|\Psi\left(X\right)\right|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon}\right)\Psi\left(X\right) \qquad (229)$$

and this depends on through K_X. Then, due to the first order condition for $\Psi\left(X\right)$, one has:

$$\frac{\delta}{\delta\hat{\Psi}}\left(S_1 + S_2\right) = \frac{\delta K_X}{\delta\hat{\Psi}}\frac{\partial}{\partial K_X}\left(S_1 + S_2\right)$$

We have assumed previously that $H\left(K_X\right)$ is slowly varying. Moreover, due to is definition:

$$\frac{\delta K_X}{\delta\hat{\Psi}\left(\hat{K}, X\right)} = \frac{\hat{K}}{\left\|\Psi\left(X\right)\right\|^2}$$

In most of the cases, this reduces to:

$$\frac{\delta K_X}{\delta\hat{\Psi}\left(\hat{K}, X\right)} \simeq \frac{\hat{K}}{D\left(\left\|\Psi\right\|^2\right)} << \hat{K}$$

As a consequence, we can assume that $\frac{\delta}{\delta\hat{\Psi}}\left(S_1 + S_2\right)$ will be negligible with respect to the other quantities in the minimization with respect to $\hat{\Psi}\left(\hat{K}, X\right)$. The rationale for this approximation is the following. The field action $S_1 + S_2$ for $\Psi\left(X\right)$ depends on the global quantity $\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}$ that represents the total investment in sector X. While minimizing the field action $S_1 + S_2$ with respect to $\hat{\Psi}\left(\hat{K}, X\right)$, we compute the change in this action with respect to an individual variation $\hat{\Psi}\left(\hat{K}, X\right)$, and the impact of this variation is, as a consequence, negligible.

A3.1.2 Minimization for $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$

Adding the Lagrange multiplier $\hat{\lambda}$ implementing the constraint $\int \left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 = \hat{N}$, the minimization of (228) with the functions given by (227) leads to the first order conditions:

$$
\begin{aligned}
0 = {} & \left(\frac{\sigma_{\hat{X}}^2 \nabla_{\hat{X}}^2}{2} - \frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{2}\right)\hat{\Psi} + \nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2 \nabla_{\hat{K}}}{2} - \hat{K}f\left(\hat{X}, K_{\hat{X}}\right) - \hat{\lambda}\right)\hat{\Psi} \qquad (230) \\
& - \left(\int \hat{\Psi}^\dagger \frac{\delta}{\delta\hat{\Psi}^\dagger}\left(\frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)\hat{\Psi}\right) - \left(\int \hat{\Psi}^\dagger \nabla_{\hat{K}} \frac{\delta}{\delta\hat{\Psi}^\dagger}\left(\hat{K}f\left(\hat{X}, K_{\hat{X}}\right)\right)\hat{\Psi}\right)
\end{aligned}
$$

Using that:

$$\frac{\delta}{\delta\hat{\Psi}^\dagger}K_{\hat{X}} = \frac{\hat{K}}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\hat{\Psi}$$

equation (230) becomes:

$$
\begin{aligned}
0 = {} & \left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 - \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)\hat{\Psi} \qquad (231) \\
& + \left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X}, K_{\hat{X}}\right)\right) - \hat{\lambda}\right)\hat{\Psi} - F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\hat{\Psi}
\end{aligned}
$$

with:

$$F\left(\hat{X}, K_{\hat{X}}\right) = \frac{\left\langle \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} + \frac{\left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} \tag{232}$$

The brackets in (232) are given by:

$$\left\langle \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle$$

$$= \int \hat{\Psi}^{\dagger}\left(\hat{X}, \hat{K}\right) \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \hat{\Psi}\left(\hat{X}, \hat{K}\right) d\hat{K}$$

$$\equiv \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2$$

$$\left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle$$

$$= \int \hat{\Psi}^{\dagger}\left(\hat{X}, K_{\hat{X}}\right) \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \right) \hat{\Psi}\left(\hat{X}, K_{\hat{X}}\right) d\hat{K}$$

$$= -\nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \int \left(\hat{K} \nabla_{\hat{K}} \left\| \hat{\Psi}\left(\hat{X}, K_{\hat{X}}\right) \right\|^2 - \frac{2\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right) \left\| \hat{\Psi}\left(\hat{X}, K_{\hat{X}}\right) \right\|^2 \right) d\hat{K}$$

$$= \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2 + \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}} \tag{233}$$

Where the average $\left\langle \hat{K}^2 \right\rangle_{\hat{X}}$ is defined by:

$$\left\langle \hat{K}^2 \right\rangle_{\hat{X}} = \int \left\| \hat{\Psi}\left(\hat{X}, \hat{K}\right) \right\|^2 d\hat{K}$$

The previous expression (233) for $F\left(\hat{X}, K_{\hat{X}}\right)$ can also be rewritten as:

$$F\left(\hat{X}, K_{\hat{X}}\right) = \frac{\left\langle \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} + \frac{\left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} \tag{234}$$

$$= \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right) \right) \frac{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2}{\left\| \Psi\left(\hat{X}\right) \right\|^2}$$

$$+ \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2 \left\| \Psi\left(\hat{X}\right) \right\|^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}}$$

It will be useful to rewrite the last term as:

$$\frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2 \left\| \Psi\left(\hat{X}\right) \right\|^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}} \simeq \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \left\langle \hat{K} \right\rangle_{\hat{X}}^2 = \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \frac{\left\| \Psi\left(\hat{X}\right) \right\|^2}{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2} \tag{235}$$

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As a consequence:

$$
\begin{aligned}
F\left(\hat{X}, K_{\hat{X}}\right) \;=\;& \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\
& + \frac{\nabla_{K_{\hat{X}}}f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}
\end{aligned}
\tag{236}
$$

We also have an equation for $\hat{\Psi}^\dagger$ similar to (231):

$$
\begin{aligned}
0 \;=\;& \left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 - \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right)\right)\hat{\Psi}^\dagger \\
& + \left(\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} + \hat{K}f\left(\hat{X}, K_{\hat{X}}\right)\right)\nabla_{\hat{K}} - \hat{\lambda}\right)\hat{\Psi} - F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\hat{\Psi}^\dagger
\end{aligned}
\tag{237}
$$

A3.1.3 Resolution of (231)

A3.1.3.1 zeroth order in σ_X^2   We consider $\sigma_X^2 << 1$ (which means that fluctuation in X << fluctuation in K). Thus (231) writes at the lowest order:

$$
\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X}, K_{\hat{X}}\right)\right) - \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right)}{2} - F\left(\hat{X}, K_{\hat{X}}\right)\hat{K} - \hat{\lambda}\right)\hat{\Psi} = 0
\tag{238}
$$

Performing the change of variable:

$$
\hat{\Psi} \to \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\hat{\Psi}
$$

leads to the equation for $\hat{K}$:

$$
\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2}f^2\left(\hat{X}\right) + F\left(\hat{X}, K_{\hat{X}}\right)\hat{K} + \frac{1}{2}f\left(\hat{X}, K_{\hat{X}}\right) + \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) + \hat{\lambda}\right)\hat{\Psi} \simeq 0
\tag{239}
$$

This equation can be normalized by dividing by $f^2\left(\hat{X}\right)$:

$$
\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}^2\hat{\Psi}}{2f^2\left(\hat{X}\right)} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} + \frac{F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}}{f^2\left(\hat{X}\right)} + \frac{\frac{f(\hat{X}, K_{\hat{X}})}{2} + \frac{(g(\hat{X}))^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) + \hat{\lambda}}{f^2\left(\hat{X}\right)}\right)\hat{\Psi} \simeq 0
$$

We then define:

$$
y = \frac{\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}{\sqrt{\sigma_{\hat{K}}^2}}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}
$$

and (231) is transformed into:

$$
\nabla_y^2\hat{\Psi} - \left(\frac{y^2}{4} + \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right)\Psi \simeq 0
\tag{240}
$$

Solutions of (240) are obtained by rewriting (240):

$$\hat{\Psi}'' + \left(p\left(\hat{X},\hat{\lambda}\right) + \frac{1}{2} - \frac{1}{4}y^2 \right)\hat{\Psi}$$

where:

$$p\left(\hat{X},\hat{\lambda}\right) = -\frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}} - \frac{1}{2} \tag{241}$$

The solution of (240) is thus:

$$\hat{\Psi}_{\hat{\lambda},C}^{(0)}\left(\hat{X},\hat{K}\right) = \sqrt{C}D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\left|f\left(\hat{X}\right)\right|\right)^{\frac{1}{2}} \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{K}}} \right) \tag{242}$$

where D_p denotes the parabolic cylinder function with parameter p and C is a normalization constant that will be computed as a function of λ using the constraint $\int \left\| \hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 = \hat{N}$.

A similar equation to (238) can be obtained for $\hat{\Psi}^\dagger$. The equivalent of (230) is (237):

$$0 = \left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2 - \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) \right)\hat{\Psi} \tag{243}$$

$$+ \left(\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} + \hat{K}f\left(\hat{X},K_{\hat{X}}\right) \right)\nabla_{\hat{K}} - \hat{\lambda} \right)\hat{\Psi} - F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi}$$

The change of variable:

$$\hat{\Psi}^\dagger \to \exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right) \right)\hat{\Psi}^\dagger$$

and the approximation $\sigma_{\hat{X}}^2 << 1$ lead ultimately to:

$$\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi}^\dagger - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2}f^2\left(\hat{X}\right) + \frac{1}{2}\nabla_{\hat{X}}f\left(\hat{X},K_{\hat{X}}\right) + \frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X}\right)\right)^2 + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) + F\left(\hat{X},K_{\hat{X}}\right) + \hat{\lambda} \right)\hat{\Psi}^\dagger \simeq 0 \tag{244}$$

which is the same equation as (239). As a consequence, the solutions of (244) write:

$$\hat{\Psi}_{\lambda,C}^{(0)\dagger}\left(\hat{X},\hat{K}\right) = \hat{\Psi}_{\lambda,C}^{(0)}\left(\hat{X},\hat{K}\right) = \sqrt{C}D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\left|f\left(\hat{X}\right)\right|\right)^{\frac{1}{2}} \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{K}}} \right) \tag{245}$$

To conclude this section, we detail the expressions for $\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}$ and $\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$. Given the expression

for $F\left(\hat{X}, K_{\hat{X}}\right)$ in (236), the term $\frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}$ arising in (242) and (245)

$$
\begin{aligned}
\frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}, K_{\hat{X}}\right)} &= \frac{\sigma_{\hat{K}}^2}{f^2\left(\hat{X}\right)} \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\
&\quad + \frac{\nabla_{K_{\hat{X}}}f^2\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}, K_{\hat{X}}\right)}\frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2} \\
&\simeq \frac{\nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}, K_{\hat{X}}\right)}\frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}
\end{aligned}
\tag{246}
$$

$\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$ arising in the definition (241) of $p\left(\hat{X}, \hat{\lambda}\right)$ is equal to:

$$
\begin{aligned}
\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} &= \frac{\sigma_{\hat{K}}^2}{2}\left(\left(\frac{\nabla_{K_{\hat{X}}}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}^2 g\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^2 f\left(\hat{X}, K_{\hat{X}}\right)}\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right. \\
&\quad \left. + 2\nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)\frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}\right)^2
\end{aligned}
\tag{247}
$$

and this simplifies as:

$$
\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \simeq 2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)\frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}\right)^2
\tag{248}
$$

since:

$$
\begin{aligned}
&\frac{\nabla_{K_{\hat{X}}}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}^2 g\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^2 f\left(\hat{X}, K_{\hat{X}}\right)}\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\
&\sim \frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}^2 g\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{K_{\hat{X}}}f\left(\hat{X}, K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^2 f\left(\hat{X}, K_{\hat{X}}\right)}\left(\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2}\right) << 1
\end{aligned}
$$

A3.1.3.2 Corrections in σ_X^2: To introduce the corrections in σ_X^2 in (231) we factor the solution as:

$$
\begin{aligned}
\hat{\Psi}_{\lambda, C}\left(\hat{K}, \hat{X}\right) &= \sqrt{C}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)D_{p\left(\hat{X}, \hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)\hat{\Psi}^{(1)}\left(\hat{K}, \hat{X}\right) \\
&\equiv \hat{\Psi}_{\lambda, C}^{(0)}\left(\hat{K}, \hat{X}\right)\hat{\Psi}^{(1)}\left(\hat{K}, \hat{X}\right)
\end{aligned}
$$

and we look for $\hat{\Psi}^{(1)}$ of the form:

$$
\hat{\Psi}^{(1)} = \exp\left(\sigma_X^2 h\left(K, X\right)\right)
\tag{249}
$$

Introducing the postulated form in (231) we are lead to:

$$\frac{\sigma_X^2}{2}\nabla_{\hat{X}}^2\left(\hat{\Psi}^{(1)}\hat{\Psi}^{(0)}_{\lambda,C}\right) + \left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi}^{(1)}\right)\hat{\Psi}^{(0)}_{\lambda,C} + \left(\nabla_{\hat{K}}\hat{\Psi}^{(1)}\right)\left(\sigma_{\hat{K}}^2\nabla_{\hat{K}}\hat{\Psi}^{(0)}_{\lambda,C} - \hat{K}f\left(\hat{X}\right)\hat{\Psi}^{(0)}_{\lambda,C}\right) = 0$$

Written in terms of $h\left(\hat{K},\hat{X}\right)$, this equation becomes at the first order in σ_X^2:

$$\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}} + \sigma_{\hat{K}}^2\nabla_{\hat{K}}^2 h\left(\hat{K},\hat{X}\right) + 2\left(\nabla_{\hat{K}}h\left(\hat{K},\hat{X}\right)\right)\left(\sigma_{\hat{K}}^2\frac{\nabla_{\hat{K}}\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}} - \hat{K}f\left(\hat{X}\right)\right) = 0 \qquad (250)$$

The solution of (250) is of the type:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right) = C\left(\hat{K},X\right)\exp\left(-2\int\left(\frac{\nabla_{\hat{K}}\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}} - \frac{\hat{K}f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right)d\hat{K}\right) = C\left(\hat{K},X\right)\exp\left(-\left(2\ln\hat{\Psi}^{(0)}_{\lambda,C} - \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\right)$$

where $C\left(X\right)$ satifies:

$$C'\left(\hat{K},X\right) = -\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}\sigma_{\hat{K}}^2}\exp\left(2\ln\hat{\Psi}^{(0)}_{\lambda,C} - \frac{\hat{K}^2f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right) = -\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}}\left(\hat{\Psi}^{(0)}_{\lambda,C}\right)^2\exp\left(-\frac{\hat{K}^2f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right)$$

and the solution of (250) is:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right) = \exp\left(-\left(2\ln\hat{\Psi}^{(0)}_{\lambda,C} - \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\right)\left(C - \int\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}\sigma_{\hat{K}}^2}\left(\hat{\Psi}^{(0)}_{\lambda,C}\right)^2\exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)d\hat{K}\right)$$

letting $C = 0$, we obtain:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right) = -\frac{1}{\sigma_{\hat{K}}^2\left(\hat{\Psi}^{(0)}_{\lambda,C}\right)^2}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\left(\int\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}}\left(\hat{\Psi}^{(0)}_{\lambda,C}\right)^2\exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)d\hat{K}\right) \quad (251)$$

To compute $h\left(K,X\right)$, we have to estimate $\frac{\nabla_{\hat{X}}^2\hat{\Psi}^{(0)}_{\lambda,C}}{\hat{\Psi}^{(0)}_{\lambda,C}}$ in (251). To do so, we write, for $\varepsilon << 1$, i.e. $\left|f\left(\hat{X}\right)\right| >> 1$:

$$\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)$$

$$\simeq \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right) - \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{4\sigma_{\hat{K}}^2}\right)\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)^{p\left(\hat{X},\hat{\lambda}\right)}$$

$$= \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right) - \frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{4\sigma_{\hat{K}}^2}\right)$$

$$\times\exp\left(\left(p\left(\hat{X},\hat{\lambda}\right)\right)\ln\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)\right)$$

which allows to compute the successives derivatives of $\hat{\Psi}$. We find, for $f > 0$:

$$
\begin{aligned}
\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} &\simeq \left(\frac{-f'\sigma_{\hat{X}}^2 \hat{\lambda} - g^2 f' + 2fgg'}{\sigma_{\hat{X}}^2 f^2} \ln\left(\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)} \right) \left(\frac{f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right) \right. \\[2mm]
&\quad + \frac{1}{2} \left(\frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2 \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda} \right)}{\sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}} + \frac{1}{2} \right) \frac{f'}{f} \\[2mm]
&\quad \left. + \frac{\hat{K}^2 - \left(\frac{\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}{2} \right)^2}{\sigma_{\hat{K}}^2} f' \right)^2 \\[2mm]
&\simeq \left(\frac{\left(4\hat{K}^2 - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)} \right)^2 \right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2
\end{aligned}
$$

The same approximation is valid for $f < 0$ and we find for this case:

$$
\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} \simeq \left(\frac{\left(4\hat{K}^2 + \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)} \right)^2 \right) f'(X)}{4\sigma_{\hat{K}}^2} \right)^2
$$

Then, introducing $\mp$ to account for the sign of $-f$, (251) becomes:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right) = -\frac{1}{\sigma_{\hat{K}}^2\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\int\left(\frac{\left(4\hat{K}^2\mp\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)}{4\sigma_{\hat{K}}^2}\right)^2 \tag{253}$$

$$\times\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2\exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)d\hat{K}$$

$$\simeq -\frac{1}{\sigma_{\hat{K}}^2\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\int\left(\frac{\left(4\hat{K}^2\mp\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)}{4\sigma_{\hat{K}}^2}\right)^2$$

$$\times\exp\left(\frac{\hat{K}^2 f\left(\hat{X}\right)-\frac{1}{2}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)d\hat{K} \tag{254}$$

$$\simeq -\frac{1}{\sigma_{\hat{K}}^2\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\int\left(\sigma_{\hat{K}}^2\frac{\left(\hat{K}^2\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)}{\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)^2}\right)^2$$

$$\times\partial_{\hat{K}}^4\exp\left(\frac{\hat{K}^2 f\left(\hat{X}\right)-\frac{1}{2}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)d\hat{K} \tag{255}$$

Assuming $\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} << 1$, we have ultimately:

$$\nabla_{\hat{K}}\left(h\left(K, X\right)\right) \simeq -\frac{1}{\sigma_{\hat{K}}^2 \left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2} \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right) \left(\sigma_{\hat{K}}^2 \frac{\left(\hat{K}^2 \mp \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)}{2\hat{K}f\left(\hat{X}\right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)\left|f\left(\hat{X}\right)\right|}\right)^2$$

$$\times \partial_{\hat{K}}^3 \exp\left(\frac{\hat{K}^2 f\left(\hat{X}\right) - \frac{1}{2}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2 \left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)$$

$$= -\frac{\left(\frac{\left(\hat{K}^2 - \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\overline{X})}\right)^2\right) f'(X)}{\sigma_{\hat{K}}^2}\right)^2}{2\hat{K}f\left(\hat{X}\right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)\left|f\left(\hat{X}\right)\right|}$$

$$= -\frac{\left(\left(\hat{K}^2 \mp \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2 \left(2\hat{K}f\left(\hat{X}\right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)\left|f\left(\hat{X}\right)\right|\right)}$$

Replacing in first approximation $\hat{K}$ by $\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\hat{N}(\hat{X})}$ in (252), and using (251) and (249) leads to:

$$\hat{\Psi}^{(1)}\left(\hat{X}\right) = \sqrt{C}\exp\left(-\int \frac{\left(\left(\hat{K}^2 \mp \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)^2\right) f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2 \left(2\hat{K}f\left(\hat{X}\right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}\right)\left|f\left(\hat{X}\right)\right|\right)} d\hat{K}\right)$$

with C a constant to be computed using the normalization condition.

To find $\Psi^\dagger$, we need also $\hat{\Psi}^{(1)\dagger}$. Writing:

$$\hat{\Psi}^{(1)\dagger} = \exp\left(\sigma_X^2 g\left(K, X\right)\right)$$

with a function $g\left(K, X\right)$ that satisfies:

$$\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}} + \sigma_{\hat{K}}^2 \nabla_{\hat{K}}^2 g\left(\hat{K}, \hat{X}\right) + 2\left(\nabla_{\hat{K}} g\left(\hat{K}, \hat{X}\right)\right)\left(\sigma_{\hat{K}}^2 \frac{\nabla_{\hat{K}} \hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}} + \hat{K}f\left(\hat{X}\right)\right) = 0$$

with:

$$\hat{\Psi}_{\lambda,C}^{(0)\dagger} = \exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right) D_{p(\hat{X}, \hat{\lambda})}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)$$

we find:

$$\nabla_{\hat{K}}\left(g\left(K, X\right)\right) = -\frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}} \exp\left(-\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right)\left(\int \frac{\nabla_{\hat{X}}^2 \hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)\dagger}}\left(\hat{\Psi}_{\lambda,C}^{(0)\dagger}\right)^2 \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right) d\hat{K}\right)$$

and:

$$\hat{\Psi}^{(1)\dagger}\left(\hat{X}\right) = \sqrt{C}\exp\left(\int \frac{\left(\left(\hat{K}^2 \pm \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)\right)^2}{\sigma_{\hat{K}}^2\left(2\hat{K}f\left(\hat{X}\right) + \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)$$

where $\pm$ accounts for the sign of f.

Ultimately, coming back to the initial definition of the fields we obtain for $\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)$ and $\hat{\Psi}_{\lambda,C}^{\dagger}\left(\hat{K},\hat{X}\right)$:

$$
\begin{aligned}
\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right) &= \sqrt{C}\exp\left(-\sigma_X^2\int\frac{\left(\left(\hat{K}^2 \mp \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right) - \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\\
&\times \exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int g\left(\hat{X}\right)d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)\\
\hat{\Psi}_{\lambda,C}^{\dagger}\left(\hat{K},\hat{X}\right) &= \sqrt{C}\exp\left(\sigma_X^2\int\frac{\left(\left(\hat{K}^2 \pm \frac{1}{4}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'\left(X\right)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right) + \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\\
&\times \exp\left(-\left(\frac{1}{\sigma_{\hat{X}}^2}\int g\left(\hat{X}\right)d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)
\end{aligned}
$$

A3.1.3.3 Computation of $\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2$ As a consequence of the previsous result, we can compute $\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2$. We start with $\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)}$. We have:

$$
\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} = C\exp\left(-\sigma_X^2\int\left(\frac{\left(\left(\hat{K}^2\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}\right.\right.
$$
$$
\left.\left.-\frac{\left(\left(\hat{K}^2\pm\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right)+\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\right)
$$

$$
= C\exp\left(-\sigma_X^2\int\left(\frac{\left(\left(\hat{K}^2-\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\sigma_{\hat{K}}^2\left(2\hat{K}\left|f\left(\hat{X}\right)\right|-\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}\right.\right.
$$
$$
\left.\left.--\frac{\left(\left(\hat{K}^2+\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}\left|f\left(\hat{X}\right)\right|+\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\right)
$$

And for $\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}<<1$:

$$
\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} \simeq C\exp\left(-\sigma_X^2\int\left(\frac{\left(\frac{3}{4}\hat{K}^2 f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\hat{K}\left|f\left(\hat{X}\right)\right|}-\frac{\left(\frac{5}{4}\hat{K}^2 f'(X)\right)^2}{3\left(\sigma_{\hat{K}}^2\right)^2\hat{K}f\left(\hat{X}\right)}\right)d\hat{K}\right)
$$

$$
= C\exp\left(-\frac{\sigma_X^2\hat{K}^4\left(f'(X)\right)^2}{96\left(\sigma_{\hat{K}}^2\right)^2\left|f\left(\hat{X}\right)\right|}\right)
$$

Gathering the previous results, we obtain the norm of $\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2$:

$$
\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2 \simeq C\exp\left(-\frac{\sigma_X^2\hat{K}^4\left(f'(X)\right)^2}{96\left(\sigma_{\hat{K}}^2\right)^2\left|f\left(\hat{X}\right)\right|}\right)D_{p(\hat{X},\hat{\lambda})}^2\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)
$$
(256)

with:

$$
f\left(\hat{X},K_{\hat{X}}\right) = \left(r\left(K_{\hat{X}},\hat{X}\right)-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^2+F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\right)\right)
$$
(257)

$$
g\left(\hat{X},K_{\hat{X}}\right) = \left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right\|}+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\right)\right)
$$
(258)

The solutions are parametrized by C and $\hat{\lambda}$ and $\hat{K}_{\hat{X}}$. Using the constraint $\left\| \hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 = \hat{N}$ will reduce the solutions to a one-parameter set of solutions. The computation of the average capital over this set will lead to the defining equation for $\hat{K}_{\hat{X}}$.

Replacing in first approximation $\hat{K}$ by its average $\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\hat{N}(\hat{X})}$ in the first term yields:

$$\left\| \hat{\Psi}_{\lambda, C}\left(\hat{K}, \hat{X}\right)\right\|^2 \simeq C \exp\left(-\frac{\sigma_X^2 \left(\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\hat{N}(\hat{X})}\right)^4 (f'(X))^2}{96\left(\sigma_{\hat{K}}^2\right)^2 \left|f\left(\hat{X}\right)\right|}\right) D^2_{p(\hat{X},\hat{\lambda})}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)$$

(259)

A3.1.4 Estimation of $S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right)$

For later purposes, we compute an estimation of $S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right)$ for any background field $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)$. We multiply (66)by $\hat{\Psi}^\dagger_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)$ on the left and integrate the equation over $\hat{K}$ and $\hat{X}$. It yields:

$$0 = S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) - \hat{\lambda}\int\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X} - \int F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X}$$

Using the constraint about the number of investors:

$$\int\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} = \hat{N}$$

we find:

$$S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) = \hat{\lambda}\hat{N} + \int F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X}$$

Moreover, equation (68) implies[26]:

$$\int F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X} \tag{260}$$

$$= \int K_{\hat{X}}\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right)\right)\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 d\hat{X}$$

$$+ \int K_{\hat{X}}\frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\left\langle\hat{K}^2\right\rangle_{\hat{X}} d\hat{X}$$

In our applications the involved functions are roughly power functions in $K_{\hat{X}}$, and as a consequence, the integral $\int F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right\|^2 d\hat{K} d\hat{X}$ is of order:

$$\int\left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right)\right)\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 d\hat{X} + \int\frac{f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\left\langle\hat{K}^2\right\rangle_{\hat{X}} d\hat{X} \quad (261)$$

---

[26] All averages in the next formula are computed in state $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)$.

Since $\left\langle \hat{K}^2 \right\rangle_{\hat{X}} \simeq K_{\hat{X}}^2 \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2}$, the second term in (??) is negligible if we assume $\frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} << 1$, i.e. the number of firms is smaller than the number of investors. As a consequence, (??) reduces to:

$$
\int \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}} \right) \right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2} \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}} \right) + f\left(\hat{X}, K_{\hat{X}} \right) \right) \left\| \hat{\Psi}\left(\hat{X} \right) \right\|^2 d\hat{X} \;\lesssim\; \int M \left\| \hat{\Psi}\left(\hat{X} \right) \right\|^2 d\hat{X}
$$

$$
= M\hat{N}
$$

where M is the lowest bound for $\left| \hat{\lambda} \right|$, computed below in (282) and (283). Our previous estimation relies on $\frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} << 1$, which is true for $f^2\left(\hat{X} \right) >> 1$. As a consequence:

$$
S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X} \right) \right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X} \right) \right) = \left(\hat{\lambda} + M \right) \hat{N} = - \left(\left| \hat{\lambda} \right| - M \right) \hat{N} \tag{262}
$$

A3.1.4 Identification of $K_{\hat{X}}$ and $N\left(\hat{X} \right)$:

A3.1.4.1 Formula depending on $\hat{\lambda}$ and C In this paragraph, we compute the average capital $K_{\hat{X}}$ and the number of investors $\hat{N}\left(\hat{X} \right)$ at $\hat{X}$ that are defined by using (192):

$$
K_{\hat{X}} \left\| \Psi\left(\hat{X} \right) \right\|^2 = \int_0^\infty \hat{K} C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}} \right)}{\left(\sigma_{\hat{K}}^2 \right)^2} \right) \tag{263}
$$

$$
\times D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}} \right)}{f^2\left(\hat{X} \right)} \right) \right) d\hat{K}
$$

and:

$$
N\left(\hat{X} \right) = C \int_0^\infty \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}} \right)}{\left(\sigma_{\hat{K}}^2 \right)^2} \right)
$$

$$
\times D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}} \right)}{f^2\left(\hat{X} \right)} \right) \right) d\hat{K}
$$

with:

$$
u\left(\hat{X}, \hat{K}_{\hat{X}} \right) = \frac{\left(\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\hat{N}(\hat{X})} \right)^4 (f'(X))^2}{96 \left| f\left(\hat{X} \right) \right|} \tag{264}
$$

Note that in these formulas, $K_{\hat{X}}$ and $N\left(\hat{X} \right)$ depend implicitly of $\hat{\lambda}$ since they have been computed in the state defined by the background field $\hat{\Psi}_{\lambda, C}\left(\hat{K}, \hat{X} \right)$. In the sequel, for the sake of simplicity, $\hat{\Psi}_{\lambda, C}\left(\hat{K}, \hat{X} \right)$, the indices λ and C may be omitted.

We will also need $\frac{K_{\hat{X}} \|\Psi(\hat{X})\|^2}{\hat{N}(\hat{X})}$ that arises in (264):

$$
\frac{K_{\hat{X}} \left\| \Psi\left(\hat{X} \right) \right\|^2}{\hat{N}\left(\hat{X} \right)} = \frac{\int_0^\infty \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right) \right) d\hat{K}}{\int_{\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}^\infty \hat{K} D_{p(\hat{X}, \hat{\lambda})}^2 \left(\left(\frac{|f(\hat{X})|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \hat{K} \right) d\hat{K}}
$$

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By a change of variable $\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \to \hat{K}$ we can also write:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \simeq C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right) \int_{\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}^{\infty} \hat{K} D_{p(\hat{X},\hat{\lambda})}^2 \left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}} \hat{K}\right) d\hat{K}$$

$$\hat{N}\left(\hat{X}\right) \simeq C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right) \int_{\frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})}}^{\infty} D_{p(\hat{X},\hat{\lambda})}^2 \left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}} \hat{K}\right) d\hat{K}$$

and by a zeroth order expansion around 0 of $\hat{K} D_{p(\hat{X},\hat{\lambda})}^2$ and $D_{p(\hat{X},\hat{\lambda})}^2$ we have:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \simeq C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right) \int_0^{\infty} \hat{K} D_{p(\hat{X},\hat{\lambda})}^2 \left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}} \hat{K}\right) d\hat{K} \qquad (265)$$

$$N\left(\hat{X}\right) \simeq C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right) \left(\int_0^{\infty} D_{p(\hat{X},\hat{\lambda})}^2 \left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}} \hat{K}\right) d\hat{K} - \frac{\left(\frac{\left|f(\hat{X})\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} 2^{\frac{p(\hat{X},\hat{\lambda})}{2}} \sqrt{\pi}}{\Gamma\left(\frac{1-p(\hat{X},\hat{\lambda})}{2}\right)} \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

$$(266)$$

To compute $N\left(\hat{X}\right)$ we use that the function D satisfies:

$$\int D_p^2 = \frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \frac{\text{Psi}\left(\frac{1}{2} - \frac{p}{2}\right) - \text{Psi}\left(-\frac{p}{2}\right)}{\Gamma\left(-p\right)}$$

The computation of the norm implies a second change of variable $\hat{K} \to \hat{K} \left(\frac{\left|f(\hat{X})\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}$ and we obtain for (266):

$$\hat{N}\left(\hat{X}\right) = \int \left\| \hat{\Psi}_{\lambda,C}\left(\hat{K}, \hat{X}\right) \right\|^2 d\hat{K} \qquad (267)$$

$$= C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right) \left(\int D_{p(\hat{X},\hat{\lambda})}^2 \left(\hat{K}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}\right) dK - \frac{\left(\frac{\left|f(\hat{X})\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} 2^{\frac{p(\hat{X},\hat{\lambda})}{2}} \sqrt{\pi}}{\Gamma\left(\frac{1-p(\hat{X},\hat{\lambda})}{2}\right)} \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

$$= C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right) \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}$$

$$\times \left(\frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \frac{\text{Psi}\left(\frac{1-p(\hat{X},\hat{\lambda})}{2}\right) - \text{Psi}\left(-\frac{p(\hat{X},\hat{\lambda})}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)} - \frac{2^{\frac{p(\hat{X},\hat{\lambda})}{2}} \sqrt{\pi}}{\Gamma\left(\frac{1-p(\hat{X},\hat{\lambda})}{2}\right)} \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

Expression (265) is computed using that:

$$\int_0^{\infty} z D_p^2\left(z\right) dz = \int_0^{\infty} D_{p+1}\left(z\right) D_p\left(z\right) dz + p \int_0^{\infty} D_{p-1}\left(z\right) D_p\left(z\right) dz$$

$$\int_0^{\infty} z D_p^2\left(z\right) dz = \frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right)\Gamma\left(-\frac{p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}$$

and:

$$\int \hat{K} D^2_{p(\hat{X},\hat{\lambda})} \left(\hat{K} \left(f^2 \left(\hat{X} \right) \right)^{\frac{1}{4}} \right) = \left(f \left(\hat{X} \right) \right)^{-1} \int u D^2_{p(\hat{X},\hat{\lambda})} (u)$$

We obtain:

$$K_{\hat{X}} \left\| \Psi \left(\hat{X} \right) \right\|^2 \simeq \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}} \right)}{16 \left(\sigma_{\hat{K}}^2 \right)^2} \right) \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{-1} C \tag{268}$$

$$\times \left(\frac{\Gamma\left(-\frac{p+1}{2} \right) \Gamma\left(\frac{1-p}{2} \right) - \Gamma\left(-\frac{p}{2} \right) \Gamma\left(\frac{-p}{2} \right)}{2^{p+2} \Gamma\left(-p-1 \right) \Gamma\left(-p \right)} + p \frac{\Gamma\left(-\frac{p}{2} \right) \Gamma\left(\frac{2-p}{2} \right) - \Gamma\left(-\frac{p-1}{2} \right) \Gamma\left(-\frac{p-1}{2} \right)}{2^{p+1} \Gamma\left(-p \right) \Gamma\left(-p+1 \right)} \right)$$

where:

$$p = p\left(\hat{X}, \hat{\lambda} \right) \tag{269}$$

Ultimately we can compute $\frac{K_{\hat{X}} \left\| \Psi(\hat{X}) \right\|^2}{\hat{N}(\hat{X})}$:

$$\frac{K_{\hat{X}} \left\| \Psi\left(\hat{X} \right) \right\|^2}{\hat{N}\left(\hat{X} \right)} \simeq \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} \frac{\frac{\Gamma\left(-\frac{p+1}{2} \right) \Gamma\left(\frac{1-p}{2} \right) - \Gamma\left(-\frac{p}{2} \right) \Gamma\left(\frac{-p}{2} \right)}{2^{p+2} \Gamma(-p-1) \Gamma(-p)} + p \frac{\Gamma\left(-\frac{p}{2} \right) \Gamma\left(\frac{2-p}{2} \right) - \Gamma\left(-\frac{p-1}{2} \right) \Gamma\left(-\frac{p-1}{2} \right)}{2^{p+1} \Gamma(-p) \Gamma(-p+1)}}{\frac{\sqrt{\pi}}{2^{\frac{3}{2}}} \frac{\mathrm{Psi}\left(\frac{1-p}{2} \right) - \mathrm{Psi}\left(-\frac{p}{2} \right)}{\Gamma(-p)}}$$

$$\equiv \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} h\left(p \right)$$

$$\simeq \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} \sqrt{p + \frac{1}{2}}$$

so that:

$$\exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}} \right)}{\left(\sigma_{\hat{K}}^2 \right)^2} \right) \simeq \exp\left(-\frac{\sigma_X^2 \left(p + \frac{1}{2} \right)^2 \left(f'(X) \right)^2}{96 \left| f\left(\hat{X} \right) \right|^3} \right) \tag{270}$$

We end this section by finding asymptotic form for $\hat{N}\left(\hat{X} \right)$ and $K_{\hat{X}} \left\| \Psi\left(\hat{X} \right) \right\|^2$

For $\varepsilon << 1$ an asymptotic form yields that:

$$D_{p(\hat{X},\hat{\lambda})} \left(\hat{K} \left(f^2 \left(\hat{X} \right) \right)^{\frac{1}{4}} \right) \simeq \exp\left(-\frac{\hat{K}^2 \left| f\left(\hat{X} \right) \right|}{4 \sigma_{\hat{K}}^2} \right) \left(\hat{K} \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right)^{p(\hat{X},\hat{\lambda})} \tag{271}$$

and we obtain:

$$\hat{N}\left(\hat{X} \right) = C \exp\left(-\frac{\sigma_X^2 u\left(\hat{X}, \hat{K}_{\hat{X}} \right)}{\sigma_{\hat{K}}^2} \right)$$

$$\times \int_0^\infty \exp\left(-\frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F(\hat{X}, K_{\hat{X}})}{f^2(\hat{X})} \right)^2 \left| f\left(\hat{X} \right) \right|}{2 \sigma_{\hat{K}}^2} \right) \left(\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}} \right)}{f^2\left(\hat{X} \right)} \right) \left(\frac{\left| f\left(\hat{X} \right) \right|}{\sigma_{\hat{K}}^2} \right)^{\frac{1}{2}} \right)^{2p(\hat{X},\hat{\lambda})} d\hat{K}$$

A change of variable $w = \dfrac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2 \left|f\left(\hat{X}\right)\right|}{2\sigma_{\hat{K}}^2}$ leads to:

$$\hat{N}\left(\hat{X}\right) \simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}\left(2^{p\left(\hat{X},\hat{\lambda}\right)-\frac{1}{2}}\Gamma\left(p\left(\hat{X},\hat{\lambda}\right)+\frac{1}{2}\right)-\frac{2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

$$(272)$$

By the same token we can use the asymptotic form (271) to find $K_{\hat{X}}$:

$$
\begin{aligned}
K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 &\simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\int \hat{K}\exp\left(-\frac{\hat{K}^2\left|f\left(\hat{X}\right)\right|}{2\sigma_{\hat{K}}^2}\right)\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)^{2p\left(\hat{X},\hat{\lambda}\right)}d\hat{K}\\
&= \frac{\sigma_{\hat{K}}^2 C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)}{\left|f\left(\hat{X}\right)\right|}\int y\exp\left(-\frac{y^2}{2}\right)y^{2p\left(\hat{X},\hat{\lambda}\right)}
\end{aligned}
$$

We set $y = \sqrt{2w}$ and we obtain:

$$
\begin{aligned}
K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 &\simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)2^{p\left(\hat{X},\hat{\lambda}\right)}\frac{\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|}\int\exp\left(-w\right)w^{p\left(\hat{X},\hat{\lambda}\right)}dw\\
&= C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)2^{p\left(\hat{X},\hat{\lambda}\right)}\frac{\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|}\Gamma\left(p\left(\hat{X},\hat{\lambda}\right)+1\right)
\end{aligned}
$$

A3.1.4.2 Computation of C as a function of $\hat{\lambda}$: Ultimately, we need to determine the value of the Lagrange multiplier $\hat{\lambda}$ and of the associated value of C. We do so by integrating (256) and the result is constrained to be $\hat{N}$, the total number of agents:

$$\hat{N} = \int\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2 d\hat{K}d\hat{X} = \int\hat{N}\left(\hat{X}\right)d\hat{X}$$

Using (267) and (270), we have:

$$
\begin{aligned}
\hat{N} &= \int\hat{N}\left(\hat{X}\right)\simeq\int C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} \qquad\qquad (273)\\
&\times\left(\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\mathrm{Psi}\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)-\mathrm{Psi}\left(-\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)}-\frac{2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)d\hat{X}\\
&\simeq\int C\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\mathrm{Psi}\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)-\mathrm{Psi}\left(-\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)}d\hat{X}
\end{aligned}
$$

with f and g given by (257) and (258). We thus obtain C as a function of $\hat{\lambda}$. For $f\left(\hat{X}\right)$ slowly varying around its average we can replace $\left|f\left(\hat{X}\right)\right|$ and $f'\left(X\right)$ by $\left\langle\left|f\left(\hat{X}\right)\right|\right\rangle$ and $\left\langle f'\left(X\right)\right\rangle$, where the bracket $\left\langle A\left(\hat{X}\right)\right\rangle$

represents the average of the quantity $A\left(\hat{X}\right)$ over the sectors space. Given that the integrated function is of order $\Gamma\left(p\right)$, we can replace the integral by the maximal values of the integrand. As a consequence, we have:

$$C\left(\bar{p}\left(\hat{\lambda}\right)\right) \simeq \frac{\exp\left(-\dfrac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(\frac{\left(\bar{p}(\hat{\lambda})+\frac{1}{2}\right)f'(X_0)}{f(\hat{X}_0)}\right)^2}{96\left|f(\hat{X}_0)\right|}\right) \hat{N}\Gamma\left(-\bar{p}\left(\hat{\lambda}\right)\right)}{\left(\dfrac{\langle|f(\hat{X})|\rangle}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}} V_r \left(\mathrm{Psi}\left(-\frac{\bar{p}(\hat{\lambda})-1}{2}\right)-\mathrm{Psi}\left(-\frac{\bar{p}(\hat{\lambda})}{2}\right)\right)} \tag{274}$$

where:

$$\bar{p}\left(\hat{\lambda}\right) = \left(-\frac{\dfrac{\left(g(\hat{X}_0)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}_0\right)+\frac{1}{2}\left|f\left(\hat{X}_0\right)\right|+\nabla_{\hat{X}}g\left(\hat{X}_0,K_{\hat{X}_0}\right)-\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}_0,K_{\hat{X}_0}\right)}{2f^2(\hat{X}_0)}+\hat{\lambda}\right)}{\left|f\left(\hat{X}_0\right)\right|}\right) \tag{275}$$

and:

$$\hat{X}_0 = \arg\min_{\hat{X}}\left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(\frac{\left(p(\hat{\lambda})+\frac{1}{2}\right)f'(X)}{f(\hat{X})}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\right) \tag{276}$$

and V_r is the volume of the reduced space where the maximum is reached defined by:

$$V_r = \sum_{\hat{X}/p(\hat{X},\hat{\lambda})=\bar{p}(\hat{\lambda})} \frac{1}{\left|\dfrac{\hat{N}''(\hat{X})}{C}\right|}$$

We thus can replace C by $C\left(\hat{\lambda}\right)$ and we are left with an infinite number of solutions of (238) parametrized by $\hat{\lambda}$ and given by (256). We write $\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2$ the solution for $\hat{\lambda}$.

A3.1.4.2 Identification equation for $K_{\hat{X}}$ To each state $\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2$, we can associate an average level of $K_{\hat{X},\hat{\lambda}}$ satisfying (268) rewritten as a function of $\hat{\lambda}$. Using (270) we find:

$$\begin{aligned} K_{\hat{X},\hat{\lambda}}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{X}\right)\right\|^2 &= \hat{K}_{\hat{X},\hat{\lambda}} \\ &= \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 \left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-1} \\ &\quad \times C\left(\bar{p}\left(\hat{\lambda}\right)\right)\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right) \end{aligned} \tag{277}$$

where:

$$p\left(\hat{X},\hat{\lambda}\right) = -\frac{\dfrac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2}+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2(\hat{X})}+\hat{\lambda}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}-\frac{1}{2} \tag{278}$$

As explained in the core of the paper, to compute $K_{\hat{X}}$ we have to average (277) over $\hat{\lambda}$ with the weight $\exp\left(-\left(S_3+S_4\right)\right)$. Given equation (231), a solution (256) for a given $\hat{\lambda}$ and taking into account the constraint $\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2=\hat{N}$, has the associated normalized weight (see (262)):

90

$$w\left(\left|\hat{\lambda}\right|\right) = \frac{\exp\left(-\left(\left|\hat{\lambda}\right| - M\right)\hat{N}\right)}{\int_{|\hat{\lambda}|>M} \exp\left(-\left(\left|\hat{\lambda}\right| - M\right)\hat{N}\right) d\left|\hat{\lambda}\right|}$$

with M is the lower bound for $\left|\hat{\lambda}\right|$.

This lower bound is found by considering (239) and adding the term proportional to $\frac{\sigma_{\hat{X}}^2}{2}$:

$$\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2\hat{\Psi} + \nabla_y^2\hat{\Psi} - \left(\left(\sqrt{f^2\left(\hat{X}\right)}\frac{y^2}{4} + \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}\right) + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)\right)\right)\Psi$$

$$(279)$$

multiplying (279) by $\hat{\Psi}^\dagger$ and integrating. It yields:

$$0 = -\frac{\sigma_{\hat{X}}^2}{2}\int\left(\nabla_{\hat{X}}\hat{\Psi}^\dagger\right)\left(\nabla_{\hat{X}}\hat{\Psi}\right) \tag{280}$$

$$-\frac{1}{2}\int\sqrt{f^2\left(\hat{X}\right)}\left(\left(\nabla_y\hat{\Psi}^\dagger\right)\left(\nabla_y\hat{\Psi}\right) + \hat{\Psi}^\dagger\frac{y^2}{4}\hat{\Psi}\right) + \int\hat{\Psi}_{y=0}^\dagger\left(\nabla_y\hat{\Psi}\right)_{y=0}$$

$$-\int\hat{\Psi}^\dagger\left(\sqrt{f^2\left(\hat{X}\right)}\frac{y^2}{4} + \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}\right) + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)\right)\Psi$$

The first part of the right hand side in (280):

$$-\frac{\sigma_{\hat{X}}^2}{2}\int\left(\nabla_{\hat{X}}\hat{\Psi}^\dagger\right)\left(\nabla_{\hat{X}}\hat{\Psi}\right) - \int\sqrt{f^2\left(\hat{X}\right)}\left(\frac{1}{2}\left(\nabla_y\hat{\Psi}^\dagger\right)\left(\nabla_y\hat{\Psi}\right) + \hat{\Psi}^\dagger\frac{y^2}{4}\hat{\Psi}\right) \tag{281}$$

includes the hamiltonian of a sum of harmonic oscillators, and thus (281) is lower than $-\frac{\int\hat{\Psi}^\dagger\sqrt{f^2(\hat{X})}\hat{\Psi}}{2}$. As a consequence, we have the inequality for all $\hat{X}$:

$$\hat{\Psi}_{y=0}^\dagger\left(\nabla_y\hat{\Psi}\right)_{y=0} + \int\hat{\Psi}^\dagger\left(\left|\hat{\lambda}\right| - \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} - \left(f\left(\hat{X}\right) + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)\Psi d\hat{K}$$

$$> \frac{\int\hat{\Psi}^\dagger\sqrt{f^2\left(\hat{X}\right)}\hat{\Psi}d\hat{K}}{2}$$

Since:

$$\left|\hat{\lambda}\right|\int|\Psi|^2 d\hat{K} = \left|\hat{\lambda}\right|\hat{N}\left(\hat{X}\right)$$

and $\hat{\Psi}_{y=0}^\dagger\left(\nabla_y\hat{\Psi}\right)_{y=0}$ is of order $1 << \hat{N}\left(\hat{X}\right)$ since it is integrated over $\hat{X}$ only. As a consequence, the condition reduces to:

$$\left|\hat{\lambda}\right|\hat{N}\left(\hat{X}\right) > \int\hat{\Psi}^\dagger\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\Psi d\hat{K}$$

that is:

$$\left|\hat{\lambda}\right| > \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$$

for each $\hat{X}$, and we have:

$$M = \max_{\hat{X}} \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2 f^2\left(\hat{X}\right)} \right) \tag{282}$$

Note that in general, for $\varepsilon << 1$, $f\left(\hat{X}\right) >> 1$ and:

$$\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2 f^2\left(\hat{X}\right)} << \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)$$

so that:

$$M \simeq \max_{\hat{X}} \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \tag{283}$$

Having found M, this yields:

$$w\left(\left|\hat{\lambda}\right|\right) = \hat{N} \exp\left(-\left(\left|\hat{\lambda}\right| - M\right)\hat{N}\right) \tag{284}$$

As a consequence, averaging equation (277) yields:

$$K_{\hat{X}} = \int K_{\hat{X},\hat{\lambda}} \hat{N} \exp\left(-\left(\left|\hat{\lambda}\right| - M\right)\hat{N}\right) d\hat{\lambda}$$

$$
\begin{aligned}
K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \;=\; & \int C\left(\hat{\lambda}\right) w\left(\left|\hat{\lambda}\right|\right) \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 (f'(X))^2}{96 \left|f\left(\hat{X}\right)\right|^3}\right) \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-1} \\
& \times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right) d\hat{\lambda}
\end{aligned}
\tag{285}
$$

with $C\left(\bar{p}\left(\hat{\lambda}\right)\right)$ given by (274). Given (284), the average value of $\left|\hat{\lambda}\right|$ is $M + \frac{1}{\hat{N}}$ and have:

$$
\begin{aligned}
K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \left|f\left(\hat{X}\right)\right| \;=\; & C\left(\bar{p}\left(-\left(M - \frac{1}{\hat{N}}\right)\right)\right) \sigma_{\hat{K}}^2 \\
& \times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)
\end{aligned}
\tag{286}
$$

with:

$$p = -\frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2 \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2 f^2(\hat{X})} - \left(M - \frac{1}{\hat{N}}\right)\right)}{\sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}} - \frac{1}{2}$$

We can consider that $\frac{1}{\hat{N}} << 1$ so that $C\left(\bar{p}\left(-\left(M - \frac{1}{\hat{N}}\right)\right)\right) \simeq C\left(\bar{p}(-M)\right)$. It amounts to consider $\left|\hat{\lambda}\right| = M$. We will also write $\bar{p}(-M) = \bar{p}$ and given (??) we have:

$$\bar{p} = \left(\frac{M - \frac{(g(\hat{X}_0))^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}_0\right) + \frac{1}{2}\left|f\left(\hat{X}_0\right)\right| + \nabla_{\hat{X}} g\left(\hat{X}_0, K_{\hat{X}_0}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}_0, K_{\hat{X}_0}\right)}{2 f^2(\hat{X}_0)}\right)}{\left|f\left(\hat{X}_0\right)\right|}\right) \tag{287}$$

and:

$$p = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X}\right) + \frac{\sqrt{f^2(\hat{X})}}{2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X}, K_{\hat{X}})}{2f^2(\hat{X})} \right) \right)}{\sqrt{f^2\left(\hat{X}\right)}} \tag{288}$$

Equation (274) rewrites:

$$C\left(\bar{p}\right) \simeq \frac{\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(\frac{\left(\bar{p}(\hat{\lambda}) + \frac{1}{2}\right) f'(X_0)}{f(\hat{X}_0)} \right)^2}{96 \left| f(\hat{X}_0) \right|} \right) \hat{N}\Gamma\left(-\bar{p}\right)}{\left(\frac{\langle |f(\hat{X})| \rangle}{\sigma_{\hat{K}}^2} \right)^{-\frac{1}{2}} V_r \left(\mathrm{Psi}\left(-\frac{\bar{p}-1}{2}\right) - \mathrm{Psi}\left(-\frac{\bar{p}}{2}\right) \right)} \tag{289}$$

and (286) reduces to:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \left| f\left(\hat{X}\right) \right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \hat{\Gamma}\left(p + \frac{1}{2} \right) \tag{290}$$

with:

$$\begin{aligned}
\hat{\Gamma}\left(p + \frac{1}{2} \right) &= \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2} \right)^2 (f'(X))^2}{96 \left| f\left(\hat{X}\right) \right|^3} \right) \\
&\times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right) \Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right) \Gamma\left(\frac{-p}{2}\right)}{2^{p+2} \Gamma\left(-p-1\right) \Gamma\left(-p\right)} + p \frac{\Gamma\left(-\frac{p}{2}\right) \Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right) \Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1} \Gamma\left(-p\right) \Gamma\left(-p+1\right)} \right)
\end{aligned} \tag{291}$$

We note that, asymptotically:

$$\hat{\Gamma}\left(p + \frac{1}{2} \right) \sim_\infty \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2} \right)^2 (f'(X))^2}{96 \left| f\left(\hat{X}\right) \right|^3} \right) \Gamma\left(p + \frac{3}{2} \right) \tag{292}$$

A3.1.4.3 Replacing $\left\| \Psi\left(X\right) \right\|^2$ in the $K_{\hat{X}}$ equation We can isolate $K_{\hat{X}}$ in (286) by using (206) and (220) to rewrite $\left\| \Psi\left(\hat{X}\right) \right\|^2$:

Using (191a):

$$\begin{aligned}
D\left(\left\| \Psi \right\|^2 \right) &= 2\tau \left\| \Psi\left(X\right) \right\|^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R\left(X\right) \right)^2 H^2 \left(\frac{\hat{K}_X}{\left\| \Psi\left(X\right) \right\|^2} \right) \left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)} \frac{\hat{K}_X}{\left\| \Psi\left(X\right) \right\|^2} \right) \\
&= 2\tau \left\| \Psi\left(X\right) \right\|^2 + \frac{1}{2\sigma_X^2} \left(\nabla_X R\left(X\right) \right)^2 H^2 \left(K_X \right) \left(1 - \frac{H'\left(K_X\right)}{H\left(K_X\right)} K_X \right)
\end{aligned}$$

We rewrite $\left\| \Psi\left(X\right) \right\|^2$ as a function of K_X:

$$\left\| \Psi\left(X\right) \right\|^2 = \frac{D\left(\left\| \Psi \right\|^2 \right) - \frac{1}{2\sigma_X^2} \left(\nabla_X R\left(X\right) \right)^2 H^2\left(K_X\right) \left(1 - \frac{H'(K_X)}{H(K_X)} K_X \right)}{2\tau} \equiv D - \bar{H}\left(X, K_X\right) \tag{293}$$

Ultimately, the equation (290) for $K_{\hat{X}}$ can be rewritten:

$$K_{\hat{X}} \left| f\left(\hat{X}\right) \right| = \frac{C\left(\bar{p}\right) \sigma_{\hat{K}}^2}{\left\| \Psi\left(X\right) \right\|^2} \hat{\Gamma}\left(p + \frac{1}{2} \right) = \frac{C\left(\bar{p}\right) \sigma_{\hat{K}}^2}{D - \bar{H}\left(X, K_X\right)} \hat{\Gamma}\left(p + \frac{1}{2} \right) \tag{294}$$

with $C\left(\bar{p}\right)$ given by (289), $\hat{\Gamma}\left(p + \frac{1}{2} \right)$ defined in (291) and p given by (288).

A3.2 Approaches to solutions for $K_{\hat{X}}$

We detail some computations of the three approaches detailed in the core of the paper.

A3.2.1 First approach: Differential form of (89)

To understand the behavior of the solutions of (89), we can write its differential version. Assume a variation $\delta Y\left(\hat{X}\right)$ for any parameter of the system at point $\hat{X}$. This parameter $Y\left(\hat{X}\right)$ can be either $R\left(X\right)$, its gradient, or any parameter arising in the definition of f and g. This induces a variation $\delta K_{\hat{X}}$ for the average capital. The equation for $\delta K_{\hat{X}}$ is obtained by differentiation of (89):

$$
\delta K_{\hat{X}} = \left(-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial \left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + l\left(\hat{X},K_{\hat{X}}\right) \right) + k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}} \right) K_{\hat{X}}\delta K_{\hat{X}} \qquad (295)
$$
$$
+ \frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} \right)\delta Y\left(\hat{X}\right)
$$

where we define:

$$
l\left(\hat{X},K_{\hat{X}}\right) = \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}
$$
$$
+ \frac{\partial p}{\partial K_{\hat{X}}}\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}
$$

$$
k\left(p\right) = \frac{\frac{d}{dp}\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\hat{\Gamma}\left(p+\frac{1}{2}\right)} \sim_\infty \sqrt{\frac{p-\frac{1}{2}}{2}} - \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3} \qquad (296)
$$

and:

$$
\frac{\partial p}{\partial K_{\hat{X}}} = \frac{\partial}{\partial K_{\hat{X}}}\frac{M - A\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|} = -\frac{\partial_{K_{\hat{X}}}\left|f\left(\hat{X},K_{\hat{X}}\right)\right|p + \partial_{K_{\hat{X}}}A\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}
$$

with:

$$
A\left(\hat{X},K_{\hat{X}}\right) = \frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X},K_{\hat{X}}\right) + \frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X},K_{\hat{X}}\right)} \right)
$$
$$
\simeq \frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X},K_{\hat{X}}\right) + \frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)
$$

In an expanded form (295) writes:

$$\delta K_{\hat{X}} = \left(k\left(p\right) \frac{\partial_{K_{\hat{X}}} \left(M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} \right.$$

$$- \left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}} \left(1 + \left(p + \mathcal{H}\left(f\left(\hat{X}, K_{\hat{X}}\right)\right) + \frac{1}{2} \right) k\left(p\right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} + l\left(\hat{X}, K_{\hat{X}}\right) \right) \right) K_{\hat{X}} \delta K_{\hat{X}}$$

$$+ \frac{\partial}{\partial Y} \left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right) 2\hat{\Gamma}\left(p + \frac{1}{2}\right)}{\left| f\left(\hat{X}, K_{\hat{X}}\right)\right| \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} \right) \delta Y$$

with $\mathcal{H}$ the heaviside function. Moreover:

$$\frac{\partial}{\partial Y} \left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right) 2\hat{\Gamma}\left(p + \frac{1}{2}\right)}{\left| f\left(\hat{X}, K_{\hat{X}}\right)\right| \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} \right) \delta Y$$

$$= \left(k\left(p\right) \frac{\partial_Y \left(M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} \right.$$

$$- \left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial Y} \left(1 + \left(p + \mathcal{H}\left(f\left(\hat{X}, K_{\hat{X}}\right)\right) + \frac{1}{2} \right) k\left(p\right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial Y}}{\left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} + m_Y\left(\hat{X}, K_{\hat{X}}\right) \right) \right) K_{\hat{X}} \delta Y$$

with:

$$m_Y\left(\hat{X}, K_{\hat{X}}\right) = \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(\nabla_Y \left(f'\left(\hat{X}\right) \right)^2 \left| f\left(\hat{X}\right)\right| - 3\left(\nabla_Y \left| f\left(\hat{X}\right)\right| \right) \left(f'\left(\hat{X}\right)\right)^2 \right) \left(p + \frac{1}{2}\right)^2}{120 \left| f\left(\hat{X}\right)\right|^4}$$

$$+ \nabla_Y p \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) \left(f'\left(X\right)\right)^2}{48 \left| f\left(\hat{X}\right)\right|^3}$$

so that:

$$\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} = \left(k\left(p\right) \frac{\partial_Y \left(M - \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} \right.$$

$$- \left(\frac{\frac{\partial f\left(\hat{X}, K_{\hat{X}}\right)}{\partial Y} \left(1 + \left(p + \mathcal{H}\left(f\left(\hat{X}, K_{\hat{X}}\right)\right) + \frac{1}{2} \right) k\left(p\right) \right)}{f\left(\hat{X}, K_{\hat{X}}\right)} + \frac{\frac{\partial \left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2}{\partial Y}}{\left\| \Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2} + m_Y\left(\hat{X}, K_{\hat{X}}\right) \right) \right) \frac{\delta Y}{D}$$

with:

$$
\begin{aligned}
D \;=\; & 1 + \left(\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}\left(1+\left(p+\mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k\left(p\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}} + l\left(\hat{X},K_{\hat{X}}\right)\right) \right. \\
& \left. -k\left(p\right)\frac{\partial_{K_{\hat{X}}}\left(M-\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)}\right)K_{\hat{X}}
\end{aligned}
$$

A3.2.2 Second approach: Expansion around particular solutions

As explained in the text, we choose to expand (290), or equivalently (294), around solutions with $p=0$.

A3.2.2.1 Particular solutions   To find the solution with $p=0$, we maximize the function:

$$
A\left(\hat{X}\right)=\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^{2}\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}
$$

We write:

$$
M=\max_{\hat{X}}A\left(\hat{X}\right) \tag{297}
$$

and denote by $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$ the solutions $\hat{X}_{M}$ of (297) with $K_{\hat{X}_{M}}$ their associated value of average capital per firm.

Given that $\hat{\Gamma}\left(\frac{1}{2}\right)=1$, (290) becomes at points $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$ and $p=0$:

$$
K_{\hat{X},M}\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|\left\|\Psi\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right\|^{2}\simeq\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)\exp\left(-\frac{\sigma_{\hat{X}}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right)^{2}}{384\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|^{3}}\right) \tag{298}
$$

This equation has in general several solutions, depending on the assumptions on $f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$. These solutions are discussed in the text. We also give the form of $K_{\hat{X}}$ for some particular form of the parameter function $f\left(\hat{X},K_{\hat{X}}\right)$.

Note that once a solution $K_{\hat{X}}$ of (294) is found, the value of $C\left(\bar{p}\right)$ can be obtained by solving (287) and using (289).

The next paragraph computes the expansion of (290) around these solutions with $p=0$. Remark that coming back to (290) and (294) for general values of p defined in (288), the value of $C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}$ can be replaced by $K_{\hat{X}_{M}}\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|\left\|\Psi\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right\|^{2}$ for any solution $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$.

A3.2.2.2 Expansion around particular solutions   We can find approximate solutions to (290):

$$
K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{299}
$$

with:

$$
\begin{aligned}
\hat{\Gamma}\left(p+\frac{1}{2}\right)\;=\;&\exp\left(-\frac{\sigma_{\hat{X}}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right) \\
&\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)
\end{aligned} \tag{300}
$$

for general form of the functions $f\left(\hat{X}\right)$ and $g\left(\hat{X}\right)$ by expanding (299), for each $\hat{X}$,around the closest point $\hat{X}_M$ satisfying (299) with $p = 0$. We use that:

$$\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right) \tag{301}$$

$$= \quad 1 - p\left(\gamma_0 + \ln 2 - 2\right) + o\left(p\right)$$

with γ_0 the Euler-Mascheroni constant, as well as the following relations:

$$\nabla_{K_{\hat{X}}}\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2 h\left(p\right)\left(f'\left(\hat{X}\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)_{p=0} \simeq -\frac{\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}$$

and:

$$\nabla_{\hat{X}}\left(-\frac{h\left(p\right)\left(f'\left(\hat{X}\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)_{p=0}$$

$$\simeq \quad -\frac{2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right| - 3\left(f'\left(\hat{X}\right)\right)^3}{120\left|f\left(\hat{X}\right)\right|^4} = \frac{f'\left(X\right)\left(3\left(f'\left(\hat{X}\right)\right)^2 - 2f''\left(X\right)\left|f\left(\hat{X}\right)\right|\right)}{120\left|f\left(\hat{X}\right)\right|^4}$$

the expansion of (299) at the lowest order, is:

$$\left(1 + \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(K_{\hat{X}} - K_{\hat{X},M}\right) + \left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)$$

$$\simeq \quad -\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X},M}}\left(K_{\hat{X}} - K_{\hat{X},M}\right)$$

$$-\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right| - 3\left(f'\left(\hat{X}\right)\right)^3}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)$$

$$-b\frac{\partial_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X},K_{\hat{X}}\right) + \frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\left(K_{\hat{X}} - K_{\hat{X},M}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}$$

$$-b\frac{\partial_{\hat{X}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X},K_{\hat{X}}\right) + \frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\left(\hat{X} - \hat{X}_M\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}$$

Given the maximization (297), the two last terms in the right hand side is equal to 0.

$$
\left(K_{\hat{X}} - K_{\hat{X},M}\right) = \frac{1}{D}\left(\sigma_X^2 \sigma_{\hat{K}}^2 \frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\right.
$$
$$
\left. - \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial \hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} - \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial \hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)
$$
$$
- \frac{1}{D}\frac{b}{2}\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right) + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)_{K_{\hat{X},M}}\left(\hat{X} - \hat{X}_M\right)
\tag{302}
$$

with:

$$
D = \left(1 + \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)} + \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2} + \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| - 3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X}_M}}
$$

and $K_{\hat{X}_M}$ solution of:

$$
K_{\hat{X},M}\left|f\left(\hat{X},K_{\hat{X},M}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X},M}\right)\right\|^2
$$
$$
\simeq \sigma_{\hat{K}}^2\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)C\left(\bar{p}\right) \simeq \sigma_{\hat{K}}^2 C\left(\bar{p}\right)
$$

The maximization condition (297) cancels the contribution due to:

$$
\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \left(f\left(\hat{X},K_{\hat{X}}\right) + \frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2} + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)
$$

To find a contribution due to this term, we have to expand (299) to the second order. The second order contributions proportional to $\left(K_{\hat{X}} - K_{\hat{X},M}\right)^2$ modifies slightly (302) and the term $\left(K_{\hat{X}} - K_{\hat{X},M}\right)\left(\hat{X} - \hat{X}_M\right)$ shifts D at the first order. Both modifications do not alter the interpretation for (302). We can thus consider the sole term:

$$
\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}
$$

Due to (294), for $H\left(K_{\hat{X}}\right)$ slowly varying, the contribution due to the derivatives of $\left\|\Psi\left(\hat{X}\right)\right\|^2$ can be neglected. Moreover the contribution due to the derivative of $\left|f\left(\hat{X}\right)\right|$ are negligible with respect to the first order terms. We can thus consider only the second order contributions due to $\hat{\Gamma}\left(p+\frac{1}{2}\right)$. In the rhs of (300), the second term is dominant. Moreover, we can check that in the second order expansion of (301), the term in p^2 can be neglected compared to $-p\left(\gamma_0 + \ln 2 - 2\right)$. As a consequence, the relevant second order correction to (302) is :

$$
b\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2 p\left(\hat{X} - \hat{X}_M\right) = b\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M - \frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2} + \frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right) + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\left(\hat{X} - \hat{X}_M\right)
$$

and the relevant contributions to (302) are:

$$
\left(K_{\hat{X}} - K_{\hat{X},M}\right) = \frac{1}{D}\left(\sigma_X^2 \sigma_{\hat{K}}^2 \frac{3\left(f'\left(\hat{X}\right)\right)^3 - 2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\right. \tag{303}
$$

$$
\left. - \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial \hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} - \frac{\frac{\partial \left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial \hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right)
$$

$$
+ \frac{1}{D}\frac{b}{2}\left(\hat{X} - \hat{X}_M\right)\nabla_{\hat{X}}^2 \left(\frac{M - \frac{\left(g(\hat{X},K_{\hat{X}})\right)^2}{\sigma_{\hat{X}}^2} + \frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right) + \nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|}\right)_{K_{\hat{X},M}} \left(\hat{X} - \hat{X}_M\right)
$$

Interpretation of (303) is given in the text.

A3.2.3 Third approach: Resolution for particular form for the functions

As stated in the text, we can find approximate solutions to (294) by choosing some forms for the parameters functions. The solutions are then studied in some ranges for average capital per firm K_X: $K_X >> 1$, $K_X >>> 1$, $K_X << 1$ and the intermediate range $\infty > K_X > 1$ In the case $K_X >>> 1$, the distinction between stable and unstable cases has to be made.

A3.2.3.1 Choice of functions f and g We can find approximate solutions to (294) if we give some particlar standard forms to the functions involved in the system. We use the form defined in the text. For $\left\|\Psi\left(X\right)\right\|^2$:

$$
\left\|\Psi\left(X\right)\right\|^2 \simeq \frac{D\left(\left\|\Psi\right\|^2\right) - \frac{F}{2\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 H\left(K_X\right)}{2\tau}
$$

we choose:

$$
H\left(K_X\right) = K_X^\eta \tag{304}
$$

so that we obtain:

$$
\left\|\Psi\left(X\right)\right\|^2 \simeq \frac{D\left(\left\|\Psi\right\|^2\right) - \frac{F}{2\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2 H\left(K_X\right) K_X^\eta}{2\tau} \equiv D - L\left(X\right)\left(\nabla_X R\left(X\right)\right)^2 K_X^\eta \tag{305}
$$

$$
f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(r\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1} - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + b\arctan\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\langle K_X^\alpha\rangle\langle R\left(X\right)\rangle} - 1\right)\right) \tag{306}
$$

$$
g\left(\hat{X},\Psi,\hat{\Psi}\right) = \nabla_{\hat{X}}R\left(\hat{X}\right)\arctan\left(\frac{K_{\hat{X}}^\alpha}{\langle K_X^\alpha\rangle}\right) + b\nabla_{\hat{X}}R\left(\hat{X}\right)\arctan\left(\frac{K_{\hat{X}}^\alpha R\left(\hat{X}\right)}{\langle K_X^\alpha\rangle\langle R\left(X\right)\rangle} - 1\right) \tag{307}
$$

and their approximation for $\frac{K_{\hat{X}}^\alpha R(\hat{X})}{\langle K_X^\alpha\rangle\langle R(X)\rangle} \simeq 1$ and $\eta = \alpha$:

$$
f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(\left(r\left(\hat{X}\right) + \frac{bR\left(\hat{X}\right)}{\left\langle K_{\hat{X}}^\alpha\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle} + \gamma L\left(\hat{X}\right)\right)K_{\hat{X}}^\alpha - \gamma D - b\right)
$$

$$
\equiv B_1\left(X\right)K_{\hat{X}}^{\alpha-1} + B_2\left(X\right)K_{\hat{X}}^\alpha - C\left(X\right)
$$

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$$g\left(\hat{X}, \Psi, \hat{\Psi}\right) \simeq K_{\hat{X}}^{\alpha} \nabla_{\hat{X}} R\left(\hat{X}\right) \left(1 + \frac{b}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle \left\langle R\left(\hat{X}\right)\right\rangle}\right) \equiv \nabla_{\hat{X}} R\left(\hat{X}\right) A\left(\hat{X}\right) K_{\hat{X}}^{\alpha}$$

$$\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right) \simeq \nabla_{\hat{X}}^{2} R\left(\hat{X}\right) \left(1 + \frac{b}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle \left\langle R\left(\hat{X}\right)\right\rangle}\right) K_{\hat{X}}^{\alpha} \equiv \nabla_{\hat{X}}^{2} R\left(\hat{X}\right) A\left(\hat{X}\right) K_{\hat{X}}^{\alpha}$$

A3.2.3.2 Solving (294)   Equation (294) can be studied by considering four cases:

Case 1 $K_{\hat{X}} \gg 1$ In that case, we assume $K_{\hat{X}}$ relatively large, but such that the approximation:

$$\left\|\Psi\left(\hat{X}\right)\right\|^{2} \simeq D \tag{308}$$

is still valid.

Equations (306) and (307) imply that the function $f\left(\hat{X}\right)$ is independent of $K_{\hat{X}}$ and that $g\left(\hat{X}\right)$ is proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$. Given (306) the function $f\left(\hat{X}\right)$ can be rewritten:

$$
\begin{aligned}
f\left(\hat{X}\right) &= \frac{1}{\varepsilon}\left(r\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^{2} + b \arctan\left(\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\left\langle K_{X}^{\alpha}\right\rangle \left\langle R\left(X\right)\right\rangle} - 1\right)\right) \\
&\simeq b\left(\frac{\pi}{2} - \frac{\left\langle K_{X}^{\alpha}\right\rangle \left\langle R\left(X\right)\right\rangle}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}\right) - \gamma D \\
&\equiv c - \frac{d}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} - \gamma D
\end{aligned}
$$

As a consequence, the expression for $f'\left(\hat{X}\right)$ is:

$$f'\left(\hat{X}\right) \simeq \frac{d \nabla_{\hat{X}} R\left(\hat{X}\right)}{K_{\hat{X}}^{\alpha} R^{2}\left(\hat{X}\right)} \tag{309}$$

Similarly, we can approximate (307) as:

$$g\left(\hat{X}\right) \simeq -\frac{\nabla_{\hat{X}} R\left(\hat{X}\right) f}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} \tag{310}$$

$$\nabla_{\hat{X}} g\left(\hat{X}\right) \simeq -\frac{\nabla_{\hat{X}}^{2} R\left(\hat{X}\right) f}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}$$

Given (308), equation (294) is:

$$K_{\hat{X}} D\left|f\left(\hat{X}\right)\right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^{2} \exp\left(-\frac{\sigma_{X}^{2} \sigma_{\hat{K}}^{2}\left(p + \frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right) \Gamma\left(p + \frac{3}{2}\right) \tag{311}$$

with:

$$p + \frac{1}{2} = \frac{M - \left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}} + \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^{2} F^{2}\left(\hat{X}, K_{\hat{X}}\right)}{2 f^{2}\left(\hat{X}\right)}\right)\right)}{\sqrt{f^{2}\left(\hat{X}\right)}}$$

Defining $V = \frac{1}{K_{\hat{X}}^{\alpha}}$, we can write (311) as an equation for $V << 1$ by replacing all quantities in term of V and then perform a first order expansion. To do so, we first, we write (311) as:

$$V - \frac{D\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 (f'(X))^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right)} = 0 \tag{312}$$

and then find an expansion in V for $\Gamma\left(p+\frac{3}{2}\right)$.

The first order expansion in V of $p+\frac{3}{2}$ is:

$$
\begin{aligned}
p+\frac{3}{2} &= \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)} \\[2mm]
&\simeq \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)}{c - \frac{d}{K_{\hat{X}}^{\alpha} R(\hat{X})}} \\[2mm]
&= \frac{M - \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X})\left(-\frac{fV}{R(\hat{X})}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}^2 R\left(\hat{X}\right)\left(-\frac{fV}{R(\hat{X})}\right)\right)}{c - \frac{dV}{R(\hat{X})}} \\[2mm]
&= \frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{R(\hat{X})}}{c} + \frac{M\frac{dV}{cR(\hat{X})}}{c}
\end{aligned}
$$

As a consequence, $\Gamma\left(p+\frac{3}{2}\right)$ arising in (317) is given by:

$$
\begin{aligned}
\Gamma\left(\frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{R(\hat{X})}}{c} + \frac{M\frac{dV}{cR(\hat{X})}}{c}\right) &= \Gamma\left(\frac{M}{c}\left(1 + \nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{MR\left(\hat{X}\right)} + \frac{dV}{cR\left(\hat{X}\right)}\right)\right) \\[2mm]
&\simeq \Gamma\left(\frac{M}{c}\right) + \frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right) f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right) \\[2mm]
&= \Gamma\left(\frac{M}{c}\right)\left(1 + \frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right) f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)
\end{aligned}
$$

Ultimately, using that at the first order:

$$\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 (f'(X))^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \simeq 1$$

equation (312) for V becomes:

$$V - \frac{D\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)\left(1 + \frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)} = 0$$

that is:

$$V - \frac{D\left(c - \frac{dV}{R(\hat{X})} - \gamma D\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)\left(1 + \frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)} = 0$$

And a first order expansion yields:

$$V - \frac{D\left(c - \gamma D\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}\left(1 - \frac{dV}{\left(c - \gamma D\right)R\left(\hat{X}\right)} - MV\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right) = 0$$

with solution:

$$V = \left(K_{\hat{X}}^{\alpha}\right)^{-1} = \frac{\frac{D(c-\gamma D)}{C(\bar{p})\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}}{1 + \frac{D(c-\gamma D)}{C(\bar{p})\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}\left(\frac{d}{(c-\gamma D)R(\hat{X})} + M\left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)}$$

Coming back to $K_{\hat{X}}^{\alpha}$ we have:

$$
\begin{aligned}
K_{\hat{X}}^{\alpha} &= \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{D\left(c - \gamma D\right)} + \frac{d}{\left(c - \gamma D\right)R\left(\hat{X}\right)} + M\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\mathrm{Psi}\left(\frac{M}{c}\right) \quad (313)\\
&= \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{D\left(c - \gamma D\right)} + \frac{d}{\left(c - \gamma D\right)R\left(\hat{X}\right)}\left(1 + M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c - \gamma D\right)}\right)\right)
\end{aligned}
$$

This solution satisfies the condition $K_{\hat{X}} \gg 1$ only if $\frac{C(\bar{p})\sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}}}{Dc} \gg 1$: formula (313) thus shows that the dependency of $K_{\hat{X}}^{\alpha}$ in the return depends on the sign of $1 + M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{M(c-\gamma D)}\right)$. If:

$$1 + M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c - \gamma D\right)}\right) > 0$$

then $K_{\hat{X}}^{\alpha}$ decreases with $R\left(\hat{X}\right)$. As stated in the text, this corresponds to an unstable equilibrium. If:

$$1 + M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{M\left(c - \gamma D\right)}\right) < 0$$

a stable equilibrium is possible and $K_{\hat{X}}^{\alpha}$ is an increasing function of $R\left(\hat{X}\right)$ and $f\left(\hat{X}\right)$. This corresponds to $\nabla_{\hat{X}}^2 R\left(\hat{X}\right) \ll 0$, which arises for instance for a maximum of $R\left(\hat{X}\right)$. In such case an increase in $R\left(\hat{X}\right)$ allows for an increased number $\left\|\Psi\left(\hat{X}\right)\right\|^2$ of firms, without reducing the average capital per firm.

Case 2 $K_{\hat{X}} \ggg 1$ **stable and unstable cases** In that case, $K_{\hat{X}} \ggg 1$, and we assume in first approximation that (discarding the factor $L\left(\hat{X}\right)$):

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D - \left(\nabla_X R\left(\hat{X}\right)\right)^2 K_{\hat{X}}^{\alpha} \ll 1 \tag{314}$$

This corresponds to a very high level of capital. Consequently, equation (306) implies that the function $f\left(\hat{X}\right)$ can be rewritten:

$$
\begin{aligned}
f\left(\hat{X}\right) &= \frac{1}{\varepsilon}\left(r\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1} - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + b\arctan\left(\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\langle K_X^{\alpha}\rangle\langle R(X)\rangle} - 1\right)\right) \\
&\simeq b\left(\frac{\pi}{2} - \frac{\langle K_X^{\alpha}\rangle\langle R(X)\rangle}{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}\right) \\
&\equiv c - \frac{d}{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)} > 0
\end{aligned}
$$

As a consequence, the expressions for $f'\left(\hat{X}\right)g\left(\hat{X}\right)$ and $\nabla_{\hat{X}}g\left(\hat{X}\right)$ (309) and (310) are still valid.

Two different cases arise in the resolution of (290).

First, we assume that $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 \neq 0$.

In this case, we will solve (290) by using (314) to replace $K_{\hat{X}} \simeq \left(\frac{D}{\left(\nabla_X R(\hat{X})\right)^2}\right)^{\frac{1}{\alpha}}$. We also change the variable $\frac{D}{\left(\nabla_X R(\hat{X})\right)^2} \to D$ temporarily for the sake of simplicity.

Inequality (314) along with $K_{\hat{X}} >>> 1$ and (306) implies that only the case $f > 0$ has to be considered.

Note that using our results about stability, it is easy to check that in that case, this solution is locally unstable. A very high level of capital has the tendency to attract more investments.

Given our assumptions, equation (126) becomes:

$$
\left(\nabla_X R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}\left(D - K_{\hat{X}}^{\alpha}\right) = C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\frac{\Gamma\left(p+\frac{3}{2}\right)}{\left|f\left(\hat{X}\right)\right|} \tag{315}
$$

or equivalently:

$$
K_{\hat{X}}^{\alpha} = D - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f(\hat{X})\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right)}{\left(\nabla_X R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}\left|f\left(\hat{X}\right)\right|} \tag{316}
$$

Then, defining $V = \frac{1}{K_{\hat{X}}^{\alpha}}$ as in the first case, we can write (316) as an equation for $V << 1$ by replacing all quantities in term of V and then perform a first order expansion.

First, we write (316) as:

$$
V - \frac{1}{D - \frac{C(\bar{p})\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f(\hat{X})\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right)}{\left(\nabla_X R(\hat{X})\right)^2 D^{\frac{1}{\alpha}}\left|f(\hat{X})\right|}} = 0 \tag{317}
$$

As in the previous case, the first order expansion in V of $\Gamma\left(p+\frac{3}{2}\right)$ arising in (317) is given by:

$$
\Gamma\left(p+\frac{3}{2}\right) \simeq \Gamma\left(\frac{M}{c}\right) + \frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right) \tag{318}
$$

Moreover, at the first order:

$$
\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \simeq 1
$$

and (317) becomes:

$$V - \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} \left|f\left(\hat{X}\right)\right|}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} \left|f\left(\hat{X}\right)\right| - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(p + \frac{3}{2}\right)} = 0$$

that is:

$$V - \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} \left(c - \frac{dV}{R(\hat{X})}\right)}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} \left(c - \frac{dV}{R(\hat{X})}\right) - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(p + \frac{3}{2}\right)} = 0 \tag{319}$$

Using (318) the first order expansion of the dominator in (319) is:

$$\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} \left(c - \frac{dV}{R\left(\hat{X}\right)}\right) - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(p + \frac{3}{2}\right)$$

$$= \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} c - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)$$

$$- \left(\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} \frac{d}{R\left(\hat{X}\right)} + \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 M}{c} \left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right) f}{MR\left(\hat{X}\right)} + \frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right)\right) V$$

so that (319) writes:

$$\frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} c}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} c - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)} \tag{320}$$

$$= \left(1 - \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} c \left(\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} \frac{d}{R(\hat{X})} + \frac{C(\bar{p})\sigma_{\hat{K}}^2 M}{c}\left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{MR(\hat{X})} + \frac{d}{cR(\hat{X})}\right)\Gamma'\left(\frac{M}{c}\right)\right)}{\left(\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} c - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)\right)^2}\right) V$$

$$+ \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} \frac{d}{R(\hat{X})}}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}} c - C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)} V$$

Equation (320) can be solved for V with solution:

$$\frac{1}{V} = D - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}} c} + \frac{d}{cR\left(\hat{X}\right)}\left(1 - \frac{\left(1 + \frac{C(\bar{p})\sigma_{\hat{K}}^2 M\Gamma\left(\frac{M}{c}\right)}{c\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}}}\left(\frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{Md} + \frac{1}{c}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)}{\left(1 - \frac{C(\bar{p})\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R(\hat{X})\right)^2 D^{1+\frac{1}{\alpha}} c}\right)}\right)$$

Ultimatly, restoring the variable:

$$D \to \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}$$

we obtain the solution $K_{\hat{X}}^{\alpha} = \frac{1}{V}$:

$$K_{\hat{X}}^{\alpha} = \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2} - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c} \tag{321}$$

$$+ \frac{d}{cR\left(\hat{X}\right)}\left(1 - \frac{\left(1 + \frac{C(\bar{p})(\nabla_{\hat{X}} R(\hat{X}))^{\frac{2}{\alpha}}\sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}}\left(\frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R(\hat{X})f}{d}\right)\Gamma'\left(\frac{M}{c}\right)\right)}{\left(1 - \frac{(\nabla_{\hat{X}} R(\hat{X}))^{\frac{2}{\alpha}} C(\bar{p})\sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}}\Gamma\left(\frac{M}{c}\right)\right)}\right)$$

As stated in the text, this is increasing in c, i.e. in $f\left(\hat{X}\right)$ and in $R\left(\hat{X}\right)$. This corresponds to a stable level of capital.

In a second case we consider that $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 \to 0$ and formula (321) is not valid anymore. Coming back to (186) leads rather to replace:

$$\left(\nabla_X R\left(X\right)\right)^2 \to \left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)} = \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}$$

If $\nabla_X^2 R\left(K_X, X\right) < 0$, (315) is replaced by:

$$K_{\hat{X}}^{\alpha}\left(D + \sigma_X^2 \left|\nabla_X^2 R\left(K_X, X\right)\right| K_{\hat{X}}^{\frac{\alpha}{2}}\right) = C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \frac{\Gamma\left(p + \frac{3}{2}\right)}{\left|f\left(\hat{X}\right)\right|}$$

with:

$$p + \frac{3}{2} \simeq \frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}\right)}$$

and the equation for K_X writes:

$$\sigma_X^2 \left|\nabla_X^2 R\left(K_X, X\right)\right| K_{\hat{X}}^{\frac{3}{2}\alpha} = \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}\right)}\right)}{\left|f\left(\hat{X}\right)\right|}$$

Since, given our assumptions $f\left(\hat{X}\right) \to c$ we find:

$$K_{\hat{X}} = \left(\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{\left|\nabla_X^2 R\left(K_X, X\right)\right| c}\Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{c}\right)\right)^{\frac{2}{3\alpha}} \tag{322}$$

Note that given (322), an equilibrium in the range $K_{\hat{X}} >>> 1$ is only possible for $c << 1$ Otherwise, there is no equilibrium for a maximum of $R\left(K_X, X\right)$. This equilibrium value of $K_{\hat{X}}$ decreases with c, which corresponds to an unstable equilibrium.

On the other hand, if $\nabla_X R\left(X\right) = 0$, expression (314) becomes:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D - \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)} K_{\hat{X}}^{\alpha} = D - \sigma_X^2 \nabla_X^2 R\left(K_X, X\right) K_{\hat{X}}^{\frac{\alpha}{2}}$$

and thus, if $\nabla_X^2 R\left(K_X, X\right) > 0$:

$$K_{\hat{X}}^{\alpha} \simeq \left(\frac{D}{\sigma_X^2 \nabla_X^2 R\left(K_X, X\right)}\right)^2 \tag{323}$$

However, this solution with $K_X >> 1$ corresponds to points such that $\nabla_X^2 R\left(K_X, X\right) > 0$ and $\nabla_X R\left(X\right) = 0$. Then, these points are minima of $R\left(X\right)$. This equilibrium may exist only if the level of capital (323) is high enough to compensate the weakness of the purely position dependent part of expected return and match the condition:

$$\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\langle K_X^{\alpha}\rangle \langle R\left(X\right)\rangle} - 1 > 0$$

Case $K_{\hat{X}} << 1$: This case is developped in the text.

Intermediate case $\infty > K_{\hat{X}} > 1$: We start with asymptotic form of (290):

$$K_{\hat{X}} \left\|\Psi\left(\hat{X}\right)\right\|^2 \left|f\left(\hat{X}\right)\right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3}\right) \Gamma\left(p + \frac{3}{2}\right) \tag{324}$$

Up to a constant that can be absorbed in the definition of $C\left(\bar{p}\right)$, we have:

$$\Gamma\left(p + \frac{3}{2}\right) \sim_{\infty} \sqrt{p + \frac{1}{2}} \exp\left(\left(p + \frac{1}{2}\right)\left(\ln\left(p + \frac{1}{2}\right) - 1\right)\right)$$

and (324) can be rewritten as:

$$K_{\hat{X}} \left\|\Psi\left(\hat{X}\right)\right\|^2 \left|f\left(\hat{X}\right)\right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{p + \frac{1}{2}} \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3} + \left(p + \frac{1}{2}\right)\left(\ln\left(p + \frac{1}{2}\right) - 1\right)\right)$$

$$\tag{325}$$

Since we are in an intermediate range for the parameters, we can replace, in first approximation, $\ln\left(p + \frac{1}{2}\right)$ by its average over this range: $\ln\left(\bar{p} + \frac{1}{2}\right)$. The exponential in (325) thus becomes:

$$\exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2} - \frac{48\left|f(\hat{X})\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2}\left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1\right)\right)^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3} + \frac{24\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X\right)\right)^2}\left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1\right)^2\right)$$

and equation (325) rewrites:

$$K_{\hat{X}} \left\|\Psi\left(\hat{X}\right)\right\|^2 \left|f\left(\hat{X}\right)\right| \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X\right)\right)^2 \exp\left(-\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2}\left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1\right)^2\right)}{96 \left|f\left(\hat{X}\right)\right|^3}\right)^{\frac{1}{4}}$$

$$= C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2} - \frac{48\left|f(\hat{X})\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2}\left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1\right)\right)^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3}\right) \sqrt{\left(p + \frac{1}{2}\right)\sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3}}}$$

$$\tag{326}$$

To solve (326) for $K_{\hat{X}}$, we proceed in two steps.

We first introduce an intermediate variable W and rewrite (326) as an equation for $K_{\hat{X}}$ and W. We set:

$$\sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X\right)\right)^2}{96 \left|f\left(\hat{X}\right)\right|^3}}\left(p + \frac{1}{2} - \frac{48\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'\left(X\right)\right)^2}\left(\ln\left(\bar{p} + \frac{1}{2}\right) - 1\right)\right) = W \tag{327}$$

and rewrite equation (326) partly in terms of W:

$$K_{\hat{X}} \left\| \Psi \left(\hat{X} \right) \right\|^2 \left(\frac{\sigma_X^2 \left(f'(X) \right)^2 \left| f \left(\hat{X} \right) \right| \exp \left(-\frac{96 \left| f(\hat{X}) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 (f'(X))^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)^2 \right)}{96 \left(\sigma_{\hat{K}}^2 \right)^3} \right)^{\frac{1}{4}}$$

$$= \ C \left(\bar{p} \right) \exp \left(-W^2 \right) \sqrt{W + 2 \sqrt{\frac{96 \left| f \left(\hat{X} \right) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'(X) \right)^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}} \qquad (328)$$

Note that, as seen from (327), W is a function of p and as such can be seen as a parameter depending on the shape of the sectors space.

Equation (328) both depends on $K_{\hat{X}}$ and W, and in a second step, we use (327) to write $K_{\hat{X}}$ as a function of W. To do so, we use that in the intermediate case $\infty > K_{\hat{X}} > 1$, we can assume that:

$$f \left(\hat{X} \right) = B_1 \left(X \right) K_{\hat{X}}^{\alpha - 1} + B_2 \left(X \right) K_{\hat{X}}^{\alpha} - C \left(\hat{X} \right) \simeq B_2 \left(X \right) K_{\hat{X}}^{\alpha} \qquad (329)$$

and that:

$$\frac{M - \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) A(\hat{X}) K_{\hat{X}}^{\alpha} \right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}^2 R \left(\hat{X} \right) A \left(\hat{X} \right) K_{\hat{X}}^{\alpha} \right)}{B_1 \left(X \right) K_{\hat{X}}^{\alpha - 1} + B_2 \left(X \right) K_{\hat{X}}^{\alpha} - C \left(\hat{X} \right)} - \frac{3}{2} \simeq \frac{M - \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) A(\hat{X}) K_{\hat{X}}^{\alpha} \right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}^2 R \left(\hat{X} \right) A \left(\hat{X} \right) K_{\hat{X}}^{\alpha} \right)}{B_2 \left(X \right) K_{\hat{X}}^{\alpha}} - \frac{3}{2}$$

$$(330)$$

Moreover, we can approximate $\left\| \Psi \left(\hat{X} \right) \right\|^2$:

$$\left\| \Psi \left(\hat{X} \right) \right\|^2 \simeq D \qquad (331)$$

Our assumptions (329), (330) and (331) allow to rewrite the relation (327) between $K_{\hat{X}}^{\alpha}$ and W as:

$$\sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2}{96 B_2^3 \left(X \right) K_{\hat{X}}^{\alpha}}} \left(p + \frac{1}{2} - \frac{48 \left| f \left(\hat{X} \right) \right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'(X) \right)^2} \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right) \right) = W$$

that is:

$$W = \sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2}{96 B_2^5 \left(X \right) K_{\hat{X}}^{3\alpha}}} \qquad (332)$$

$$\times \left(M - \left(\frac{\left(\nabla_{\hat{X}} R \left(\hat{X} \right) \right)^2 A \left(\hat{X} \right)}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4 \left(X \right) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) K_{\hat{X}}^{2\alpha} - \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2 \left(X \right) \right) K_{\hat{X}}^{\alpha} \right)$$

To solve this equation for $K_{\hat{X}}^{\alpha}$, we consider M as the dominant parameter and find an approximate solution of (332). At the lowest order, we write:

$$\sqrt{\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2}{96 B_2^5 \left(X \right) K_{\hat{X}}^{3\alpha}}} M = W$$

with solution:

$$K_{\hat{X}}^{\alpha} = \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5 \left(X \right) W^2} \right)^{\frac{1}{3}}$$

Considering corrections to this result, the solution to (332) is decomposed as:

$$K_{\hat{X}}^{\alpha} = \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} + \chi \tag{333}$$

and using the following intermediate results:

$$K_{\hat{X}}^{2\alpha} = \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} \left(1 + 2\chi \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} \right)$$

$$K_{\hat{X}}^{3\alpha} = \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right) \left(1 + 3\chi \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} \right)$$

we are lead to rewrite (332) as an equation for χ at first order:

$$\chi \left(\frac{3}{2} \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{-\frac{1}{3}} W \right.$$

$$+ 2\frac{W}{M} \left(\frac{\left(\nabla_{\hat{X}} R \left(\hat{X} \right) \right)^2 A \left(\hat{X} \right)}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} + \frac{W}{M} \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right)$$

$$= -\frac{W}{M} \left(\frac{\left(\nabla_{\hat{X}} R \left(\hat{X} \right) \right)^2 A \left(\hat{X} \right)}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}$$

$$- \frac{W}{M} \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}}$$

whose solution is:

$$\chi = -\frac{\left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right) + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)}$$

so that (333) yields $K_{\hat{X}}^{\alpha}$:

$$K_{\hat{X}}^{\alpha} - \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{1}{3}} \tag{3}$$

$$= -\frac{\left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right) + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)}$$

$$\frac{\left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right) + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}}}{\frac{3}{2} M + 2 \left(\frac{\left(\nabla_{\hat{X}} R(\hat{X}) \right)^2 A(\hat{X})}{\sigma_{\hat{X}}^2} + \frac{48 B_2^4(X) \left(\ln \left(\bar{p} + \frac{1}{2} \right) - 1 \right)}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2} \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)^{\frac{2}{3}} + \left(\nabla_{\hat{X}}^2 R \left(\hat{X} \right) + B_2(X) \right) \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'(X) \right)^2 M^2}{96 B_2^5(X) W^2} \right)}$$

In a third step, we can use equation (334) to rewrite (328) in an approximate form. Actually, expression (334) implies that in the intermediate case, where $K_{\hat{X}}^{\alpha}$ is of finite magnitude, we have $W^2 \sim \sigma_X^2 \sigma_{\hat{K}}^2 M^2$ and:

$$\exp\left(-W^2 + \frac{24\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

$$\simeq \exp\left(\frac{24\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

Moreover using that:

$$W + 2\sqrt{\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2\,(f'(X))^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right) \simeq 2\sqrt{\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2\,(f'(X))^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)$$

and that ultimately the left hand side of equation (328) writes at the first order:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left(\frac{\sigma_X^2\,(f'(X))^2\left|f\left(\hat{X}\right)\right|\exp\left(-\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2\,(f'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}$$

$$= \left(\frac{\sigma_X^2\,(B_2'(X))^2\,|B_2(X)|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\left\|\Psi\left(\hat{X}\right)\right\|^2\exp\left(-\frac{24\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2\,(f'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

$$\simeq D\left(\frac{\sigma_X^2\,(B_2'(X))^2\,|B_2(X)|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\exp\left(-\frac{24\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

equation (328) writes:

$$D\left(\frac{\sigma_X^2\,(B_2'(X))^2\,|B_2(X)|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\exp\left(-\frac{24\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

$$= C\left(\bar{p}\right)\sqrt{2\sqrt{\frac{96\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}$$

that is:

$$K_{\hat{X}}^{1+\frac{\alpha}{2}}\exp\left(-\frac{24\,|B_2(X)|^3\,K_{\hat{X}}^{\alpha}}{\sigma_X^2 \sigma_{\hat{K}}^2\,(B_2'(X))^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right) = \frac{8C\left(\bar{p}\right)}{D}\sqrt{\frac{3\sigma_{\hat{K}}^2\,|B_2(X)|}{\sigma_X^2\,(B_2'(X))^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right) \quad (335)$$

Equation (335) has the form:

$$x^d \exp\left(-ax\right) = c$$

with solution:

$$x = c^{\frac{1}{d}}\exp\left(-W_0\left(-\frac{a}{d}c^{\frac{1}{d}}\right)\right)$$

where W_0 is the Lambert W function with parameter 0. Applying this result to our case with:

$$
\begin{aligned}
d &= \frac{1+\alpha}{2\alpha} \\
x &= K_{\hat{X}}^{\alpha} \\
a &= \frac{24\left|B_2\left(X\right)\right|^3}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'\left(X\right)\right)^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2 \\
c &= \frac{8C\left(\bar{p}\right)}{D} \sqrt{\frac{3\sigma_{\hat{K}}^2 \left|B_2\left(X\right)\right|}{\sigma_X^2 \left(B_2'\left(X\right)\right)^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}
\end{aligned}
$$

we obtain:

$$
\begin{aligned}
K_{\hat{X}}^{\alpha} =\ & \left(\frac{8C\left(\bar{p}\right)}{D} \sqrt{\frac{3\sigma_{\hat{K}}^2 \left|B_2\left(X\right)\right|}{\sigma_X^2 \left(B_2'\left(X\right)\right)^2} \left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}\right)^{\frac{2\alpha}{1+\alpha}} \\
& \times \exp\left(-W_0\left(-\frac{48\alpha}{1+\alpha}\left(\sqrt{\frac{3\sigma_{\hat{K}}^2}{\sigma_X^2}}\frac{8C\left(\bar{p}\right)}{D}\right)^{\frac{2\alpha}{1+\alpha}} \frac{\left|B_2\left(X\right)\right|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B_2'\left(X\right)\right)^{2+\frac{2\alpha}{1+\alpha}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2+\frac{\alpha}{1+\alpha}}\right)\right)
\end{aligned}
$$

As stated in the text, this is an increasing function of $B_2\left(X\right)$. Moreover, the corrections to this formula, given in (334) show that $K_{\hat{X}}^{\alpha}$ is a decreasing function of $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$.

Appendix 4. Dynamics for $K_{\hat{X}}$

A4.1 Variation of the defining equation for $K_{\hat{X}}$

A4.1.1 Compact formulation

As claimed in the text, we consider the dynamics for $K_{\hat{X}}$ generated by modification of the parameters. To do so, we compute the variation of equation (294). We need the variations of the functions involved in (294) with respect to two dynamical variables $K_{\hat{X}}$ and $R\left(X\right)$. Starting with (294):

$$
K_{\hat{X}}\left(D - L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\right) = \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|} \hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{336}
$$

where:

$$
\begin{aligned}
\hat{\Gamma}\left(p+\frac{1}{2}\right) =\ & \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \\
& \times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)
\end{aligned} \tag{337}
$$

We first compute the variations of the right hand side and use that, in first approximation:

$$
\frac{d}{dp}\left(\ln\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)\right) \simeq \ln\left(p+\frac{1}{2}\right) \tag{338}
$$

and:

$$
\frac{d}{dp}\left(-\frac{\sigma_X^2 \left(p+\frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) = -\frac{\sigma_X^2 \left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}
$$

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so that:

$$\frac{\frac{d}{dp}\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\hat{\Gamma}\left(p+\frac{1}{2}\right)} \simeq \ln\left(p+\frac{1}{2}\right) - \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)(f'(X))^2}{48\left|f\left(\hat{X}\right)\right|^3} \tag{339}$$

Assuming that $C\left(\bar{p}\right)$ is constant, (339) allows to rewrite the variation of of equation (336):

$$\nabla_\theta\left(K_{\hat{X}}\left(D - L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)\right) = K_{\hat{X}}\left(D - L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)$$

$$\times\left(-\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} + \left(\ln\left(p+\frac{1}{2}\right) - \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)(f'(X))^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p\right)$$

$$+\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2(f'(X))^2}{96\left|f\left(\hat{X}\right)\right|^3}\left(\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} - \frac{\nabla_\theta\left(\frac{f'(X)}{f(\hat{X})}\right)^2}{\left(\frac{f'(X)}{f(\hat{X})}\right)^2}\right)$$

and we deduce from this equation, that the dynamic version of equation (336) is:

$$\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - \frac{\nabla_\theta\left(L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)}{D - L\left(\hat{X}\right)K_{\hat{X}}^\eta} = \left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)(f'(X))^2}{48\left|f\left(\hat{X}\right)\right|^3} - 1\right)\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} \tag{340}$$

$$+\left(\ln\left(p+\frac{1}{2}\right) - \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)(f'(X))^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p$$

$$-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\left(\nabla_\theta\left(\frac{f'(X)}{f\left(\hat{X}\right)}\right)^2\right)$$

A4.1.2 Expanded form of (340)

To find the dynamic equation for $K_{\hat{X}}$ we expand each side of (340).

The left hand side of (340) can be developped as:

$$\left(1 - \eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{D - L\left(\hat{X}\right)K_{\hat{X}}^\eta}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - \frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{D - L\left(\hat{X}\right)K_{\hat{X}}^\eta}\frac{\nabla_\theta L\left(\hat{X}\right)}{L\left(\hat{X}\right)}$$

$$= \left(1 - \eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - \frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta L\left(\hat{X}\right)}{L\left(\hat{X}\right)}$$

$$= \left(1 - \eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - \frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta\left((\nabla_X R(X))^2 + \sigma_X^2\frac{\nabla_X^2 R(K_X,X)}{H(K_X)}\right)}{(\nabla_X R(X))^2 + \sigma_X^2\frac{\nabla_X^2 R(K_X,X)}{H(K_X)}}$$

$$\simeq \left(1 - \eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - 2\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta\left(\nabla_X R(X)\right)}{\nabla_X R(X)}$$

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and (340) becomes:

$$\left(1 - \eta \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - 2\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \frac{\nabla_X\left(\nabla_\theta R\left(X\right)\right)}{\nabla_X R\left(X\right)} \tag{341}$$

$$= \left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3} - 1\right) \frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} + \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right) \nabla_\theta p$$

$$- \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|} \nabla_\theta\left(\frac{f'\left(X\right)}{f\left(\hat{X}\right)}\right)^2$$

To compute the right hand side of (341). We use that:

$$p = -\frac{M - \left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)\right)}{\sigma_{\hat{X}}^2 f\left(\hat{X}\right)} - \frac{3}{2}$$

so that, the variation $\nabla_\theta p$ is given by:

$$\nabla_\theta p = -\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\left(p + \frac{3}{2}\right) - \left(2\frac{g\left(\hat{X}\right)\nabla_\theta g\left(\hat{X}\right)}{\sigma_{\hat{X}}^2} + \nabla_\theta\nabla_{\hat{X}} g\left(\hat{X}\right)\right)$$

To compute $\nabla_\theta p$ we have to use the form of the functions defined in Appendix 2. We thus obtain:

$$\frac{\nabla_\theta g\left(\hat{X}\right)}{g\left(\hat{X}\right)} = \frac{\nabla_\theta\nabla_{\hat{X}} R\left(\hat{X}\right)}{\nabla_{\hat{X}} R\left(\hat{X}\right)} + \alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}$$

$$\frac{\nabla_\theta\nabla_{\hat{X}} g\left(\hat{X}\right)}{\nabla_{\hat{X}} g\left(\hat{X}\right)} = \frac{\nabla_\theta\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} + \alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}$$

and as a consequence:

$$\nabla_\theta p = -\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\left(p + \frac{3}{2}\right) - \left(2\frac{g^2\left(\hat{X}\right)\left(\frac{\nabla_\theta\nabla_{\hat{X}} R(\hat{X})}{\nabla_{\hat{X}} R(\hat{X})} + \alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)}{\sigma_{\hat{X}}^2\left|f\left(\hat{X}\right)\right|} + \frac{\nabla_{\hat{X}} g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\left(\frac{\nabla_\theta\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} + \alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)\right)$$

Ultimately, the right hand side of (341) is given by:

$$\left(\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) (f'(X))^2}{48 \left|f\left(\hat{X}\right)\right|^3} - 1 \right) \frac{\nabla_\theta \left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} + \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) (f'(X))^2}{48 \left|f\left(\hat{X}\right)\right|^3} \right) \nabla_\theta p$$

$$- \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2}{96 \left|f\left(\hat{X}\right)\right|} \nabla_\theta \left(\frac{f'(X)}{f\left(\hat{X}\right)} \right)^2$$

$$= -\frac{\nabla_\theta \left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} \left(\left(1 - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 (f'(X))^2}{96 \left|f\left(\hat{X}\right)\right|^3} \right) + \left(p + \frac{3}{2}\right) \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) (f'(X))^2}{48 \left|f\left(\hat{X}\right)\right|^3} \right) \right)$$

$$- \left(2 \frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2 \left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta \nabla_{\hat{X}} R\left(\hat{X}\right)}{\nabla_{\hat{X}} R\left(\hat{X}\right)} + \frac{\nabla_{\hat{X}} g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta \nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} + \frac{\alpha \left(2 \frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right) \right)}{\left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \right)$$

$$\times \left(\ln\left(p + \frac{1}{2}\right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right) (f'(X))^2}{48 \left|f\left(\hat{X}\right)\right|^3} \right) - \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2}{96 \left|f\left(\hat{X}\right)\right|} \nabla_\theta \left(\frac{f'(X)}{f\left(\hat{X}\right)} \right)^2$$

so that the variational equation for $K_{\hat{X}}$ (341) writes:

$$\left(1 - \eta \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \right) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} - 2 \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \frac{\nabla_X \left(\nabla_\theta R(X)\right)}{\nabla_X R(X)} \tag{342}$$

$$= -C_3\left(p, \hat{X}\right) \frac{\nabla_\theta \left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} - C_1\left(p, \hat{X}\right) \frac{\nabla_\theta \left(\frac{f'(X)}{f(\hat{X})} \right)^2}{\left(\frac{f'(X)}{f(\hat{X})} \right)^2}$$

$$- C_2\left(p, \hat{X}\right) \left(2 \frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2 \left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta \nabla_{\hat{X}} R\left(\hat{X}\right)}{\nabla_{\hat{X}} R\left(\hat{X}\right)} + \frac{\nabla_{\hat{X}} g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta \nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} + \frac{\alpha \left(2 \frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right) \right)}{\left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \right)$$

with:

$$C_1\left(p, \hat{X}\right) = \frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p + \frac{1}{2}\right)^2 (f'(X))^2}{96 \left|f\left(\hat{X}\right)\right|^3} \tag{343}$$

$$C_2\left(p, \hat{X}\right) = \ln\left(p + \frac{1}{2}\right) - \frac{2 C_1\left(p, \hat{X}\right)}{p + \frac{1}{2}}$$

$$C_3\left(p, \hat{X}\right) = 1 - C_1\left(p, \hat{X}\right) + \left(p + \frac{3}{2}\right) C_2\left(p, \hat{X}\right)$$

These term can be reordered and the general dynamic equation for $K_{\hat{X}}$ is ultimately written as:

$$\left(1 - \eta \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} + \frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right)}{\left|f\left(\hat{X}\right)\right|} C_2\left(p, \hat{X}\right)\right) \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} \tag{344}$$

$$+ 2\left(\frac{g^2\left(\hat{X}\right) C_2\left(p, \hat{X}\right)}{\sigma_{\hat{X}}^2 \left|f\left(\hat{X}\right)\right|} - \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right) \frac{\nabla_\theta \nabla_{\hat{X}} R\left(\hat{X}\right)}{\nabla_{\hat{X}} R\left(\hat{X}\right)} + \frac{\nabla_{\hat{X}} g\left(\hat{X}\right) C_2\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\nabla_\theta \nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}$$

$$= -C_3\left(p, \hat{X}\right) \frac{\nabla_\theta \left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|} - C_1\left(p, \hat{X}\right) \frac{\nabla_\theta \left(\frac{f'(X)}{f(\hat{X})}\right)^2}{\left(\frac{f'(X)}{f(\hat{X})}\right)^2}$$

A4.1.3 Dynamic equation for particular forms of $f\left(\hat{X}, K_{\hat{X}}\right)$ and $\left\|\Psi\left(\hat{X}\right)\right\|^2$

We can put equation (344) in a specific form, by using the explicit formula for $f\left(\hat{X}, K_{\hat{X}}\right)$ and $\left\|\Psi\left(\hat{X}\right)\right\|^2$ given in appendix 2. We have:

$$\frac{\nabla_\theta f\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}, K_{\hat{X}}\right)} \simeq \frac{r\left(K_{\hat{X}}, \hat{X}\right)\left(\frac{\nabla_\theta r\left(K_{\hat{X}}, \hat{X}\right)}{r\left(K_{\hat{X}}, \hat{X}\right)} + (\alpha - 1)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)}{f\left(\hat{X}\right)}$$

$$+ \frac{\gamma\left(\eta L\left(\hat{X}\right) K_{\hat{X}}^\eta \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + 2L\left(\hat{X}\right) K_{\hat{X}}^\eta \frac{\nabla_\theta(\nabla_X R(X))}{\nabla_X R(X)}\right) + F_1'\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \frac{\nabla_\theta R\left(K_{\hat{X}}, \hat{X}\right)}{R\left(K_{\hat{X}}, \hat{X}\right)}}{f\left(\hat{X}\right)}$$

$$\simeq \frac{r\left(K_{\hat{X}}, \hat{X}\right)\left(\frac{\nabla_\theta r\left(\hat{X}\right)}{r\left(K_{\hat{X}}, \hat{X}\right)} + (\alpha - 1)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)}{f\left(\hat{X}\right)}$$

$$+ \frac{\gamma\left(\eta L\left(\hat{X}\right) K_{\hat{X}}^\eta \frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + 2L\left(\hat{X}\right) K_{\hat{X}}^\eta \frac{\nabla_\theta(\nabla_X R(X))}{\nabla_X R(X)}\right) + \varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \frac{\nabla_\theta R\left(K_{\hat{X}}, \hat{X}\right)}{R\left(K_{\hat{X}}, \hat{X}\right)}}{f\left(\hat{X}\right)}$$

$$= \frac{r\left(K_{\hat{X}}, \hat{X}\right) \frac{\nabla_\theta r\left(\hat{X}\right)}{r\left(K_{\hat{X}}, \hat{X}\right)}}{f\left(\hat{X}\right)}$$

$$+ \frac{\left(\gamma\eta L\left(\hat{X}\right) K_{\hat{X}}^\eta + (\alpha - 1)\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + \varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \frac{\nabla_\theta R\left(K_{\hat{X}}, \hat{X}\right)}{R\left(K_{\hat{X}}, \hat{X}\right)} + 2\gamma L\left(\hat{X}\right) K_{\hat{X}}^\eta \frac{\nabla_\theta(\nabla_X R(X))}{\nabla_X R(X)}}{f\left(\hat{X}\right)}$$

To compute $\nabla_\theta \ln\left(\frac{f'(X)}{f(\hat{X})}\right)^2$ arising in (344), we use that in first approximation, for relatively large $K_{\hat{X}}$:

$$\left(\frac{f'(X)}{f\left(\hat{X}\right)}\right)^2 \simeq \left(\frac{F_1'\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \nabla_{\hat{X}} R\left(K_{\hat{X}}, \hat{X}\right)}{F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}\right)^2 \simeq \left(\varsigma \nabla_{\hat{X}} R\left(K_{\hat{X}}, \hat{X}\right)\right)^2$$

that can be considered int the sequel negligible at the first order.

As a consequence, for the chosen forms of the parameter functions, the dynamics equation (344) becomes ultimately:

$$k\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}} + l\frac{\nabla_\theta R\left(\hat{X}\right)}{R\left(\hat{X}\right)} - 2m\frac{\nabla_{\hat{X}}\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)} + n\frac{\nabla_{\hat{X}}^2\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)} = -C_3\left(p,\hat{X}\right)\frac{\nabla_\theta r\left(\hat{X}\right)}{f\left(\hat{X}\right)} \tag{345}$$

with:

$$
\begin{aligned}
k &= 1 - \eta\left(1 - \frac{\gamma C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\[2mm]
&\quad + \frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right) - (1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\[4mm]
l &= \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)} \\[4mm]
m &= \left(1 - \frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} - \frac{g^2\left(\hat{X}\right)C_2\left(p,\hat{X}\right)}{\sigma_{\hat{X}}^2} \\[4mm]
n &= \frac{\nabla_{\hat{X}}g\left(\hat{X}\right)C_2\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}
\end{aligned}
\tag{346}
$$

A4.2 Full dynamical system

To make the system self-consistent, we introduce also a dynamics for R.

We assume that R depends on $K_{\hat{X}}$, $\hat{X}$ and $\nabla_\theta K_{\hat{X}}$, that leads to write: $R\left(K_{\hat{X}}, \hat{X}, \nabla_\theta K_{\hat{X}}\right)$. The variation is assumed to follow a diffusion process:

$$
\begin{aligned}
\nabla_\theta R\left(\theta,\hat{X}\right) &= \int_{\theta'<\theta} G_1\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta'}R\left(\theta',\hat{X}'\right)d\left(\theta',\hat{X}'\right) \\
&\quad + \int_{\theta'<\theta} G_2\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta'}K_{\hat{X}'}d\left(\theta',\hat{X}'\right)
\end{aligned}
$$

The first orders expansion of the right hand side leads to the following form for $\nabla_\theta R\left(\theta,\hat{X}\right)$:

$$
\begin{aligned}
\nabla_\theta R\left(\theta,\hat{X}\right) &= \int\left(\hat{X} - \hat{X}'\right)\left(G_1\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}\nabla_\theta R\left(\theta,\hat{X}\right) + G_2\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}\nabla_\theta K_{\hat{X}}\right) \\
&\quad + \frac{1}{2}\int\left(\hat{X} - \hat{X}'\right)^2\left(G_1\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}^2\nabla_\theta R\left(\theta,\hat{X}\right) + G_2\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}^2\nabla_\theta K_{\hat{X}}\right) \\
&\quad + \int\left(\theta - \theta'\right)\left(G_1\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_\theta\nabla_\theta R\left(\theta,\hat{X}\right) + G_2\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_\theta\nabla_\theta K_{\hat{X}}\right) \\
&\quad + \frac{1}{2}\int\left(\theta - \theta'\right)^2\left(G_1\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_\theta^2\nabla_\theta R\left(\theta,\hat{X}\right) + G_2\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_\theta^2\nabla_\theta K_{\hat{X}}\right) \\
&\quad + ...
\end{aligned}
\tag{347}
$$

where the crossed derivatives have been discarded for the sake of simplicity. We assume $G_1\left(\left(\theta,\hat{X}\right),\left(\theta,\hat{X}\right)\right) = 0$ to avoid auto-interaction.

Performing the integrals yields:

$$\nabla_\theta R\left(\theta, \hat{X}\right) = a_0\left(\hat{X}\right)\nabla_\theta K_{\hat{X}} + a\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta K_{\hat{X}} + b\left(\hat{X}\right)\nabla_{\hat{X}}^2\nabla_\theta K_{\hat{X}}$$

$$+ c\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right) + d\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta K_{\hat{X}}\right)$$

$$+ e\left(\hat{X}\right)\nabla_{\hat{X}}\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right) + f\left(\hat{X}\right)\nabla_{\hat{X}}^2\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right)$$

$$+ g\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right) + h\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right)$$

$$+ u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right) + v\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta R\left(\theta, \hat{X}\right)\right) \tag{348}$$

We ssume that the coefficients are slowly varying, since their are obtained by averages.

Gathering the dynamics (345) and (348) for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R\left(\theta, \hat{X}\right)$ leads to a matricial system:

$$0 = \begin{pmatrix} \frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} \\ -a_0\left(\hat{X}\right) & 1 \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} \tag{349}$$

$$- \begin{pmatrix} 0 & \frac{2m}{\nabla_{\hat{X}}R(\hat{X})}\nabla_{\hat{X}} \\ a\left(\hat{X}\right)\nabla_{\hat{X}} + c\left(\hat{X}\right)\nabla_\theta & e\left(\hat{X}\right)\nabla_{\hat{X}} + g\left(\hat{X}\right)\nabla_\theta \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix}$$

$$- \begin{pmatrix} 0 & -\frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})}\nabla_{\hat{X}}^2 \\ d\left(\hat{X}\right)\nabla_\theta^2 + b\left(\hat{X}\right)\nabla_{\hat{X}}^2 + u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta & e\left(\hat{X}\right)\nabla_\theta^2 + f\left(\hat{X}\right)\nabla_{\hat{X}}^2 + v\nabla_{\hat{X}}\nabla_\theta \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix}$$

A4.3 Oscillatory solutions

We look for a solution of (350) of the form:

$$\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\left(\hat{X}\right) \end{pmatrix} = \exp\left(i\Omega\left(\hat{X}\right)\theta + iG\left(\hat{X}\right)\hat{X}\right)\begin{pmatrix} \nabla_\theta K_0 \\ \nabla_\theta R_0 \end{pmatrix}$$

with $G\left(\hat{X}\right)$ and $\Omega\left(\hat{X}\right)$ slowly varying. As a consequence, the system (349) writes:

$$\begin{pmatrix} \frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} - i\frac{2m}{\nabla_{\hat{X}}R(\hat{X})}G - \frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})}G^2 \\ -a_0\left(\hat{X}\right) - ia\left(\hat{X}\right)G - ic\left(\hat{X}\right)\Omega & 1 - ie\left(\hat{X}\right)G - ig\left(\hat{X}\right)\Omega + e\Omega^2 \\ + d\Omega^2 + bG^2 + u\Omega G & + fG^2 + u\Omega G \end{pmatrix}\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R \end{pmatrix} = 0 \tag{350}$$

By canceling the determinant of the system, we are led to the following relation between $\Omega\left(\hat{X}\right)$ and $G\left(\hat{X}\right)$:

$$0 = \frac{k}{K_{\hat{X}}}\left(1 - ieG - ig\Omega\right) + \left(\frac{l}{R\left(\hat{X}\right)} - i\frac{2m}{\nabla_{\hat{X}}R\left(\hat{X}\right)}G\right)\left(a_0 + iaG + ic\Omega\right)$$

$$- \frac{l}{R\left(\hat{X}\right)}\left(d\Omega^2 + bG^2 + u\Omega G\right) + \frac{k}{K_{\hat{X}}}\left(e\Omega^2 + fG^2 + v\Omega G\right)$$

In the sequel, we restrict to the first order terms, which yields the expression for Ω:

$$
\Omega = \frac{i}{\left(\frac{lc}{R(\hat{X})} - i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right) - \frac{kg}{K_{\hat{X}}}} \left(\frac{k}{K_{\hat{X}}}(1 - ieG) + \left(\frac{l}{R(\hat{X})} - i\frac{2m}{\nabla_{\hat{X}}R(\hat{X})}G\right)(a_0 + iaG)\right)
$$

$$
= \frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right) + i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}
$$

$$
\times \left(\left(\left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})}\right)\right)G + i\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2\right)\right)\right)
$$

Or equivalently:

$$
\Omega = \frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right)\left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})}\right)\right)G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2\right)}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}
$$

$$
+i\frac{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right)\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})} + \frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2\right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{ke}{K_{\hat{X}}} + \left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})} - \frac{la}{R(\hat{X})}\right)\right)G^2}{\left(\frac{lc}{R(\hat{X})} - \frac{kg}{K_{\hat{X}}}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}
$$

We focus on the influence of time variations of $\nabla_\theta K_{\hat{X}}$ on $\nabla_\theta R$, and we can assume $g \simeq 0$ so that there is no self influence of $\nabla_\theta R$ on itself: $\nabla_\theta R$ depends on the variations of $\nabla_\theta K_{\hat{X}}$ as well as the neighboorhood sectors variations of $\nabla_\theta R$. Moreover, the coefficients e and a, being obtained by integration or first order expansion, can be considered as nul.

As a consequence, the equation for Ω reduces to:

$$
\Omega = \frac{\frac{lc}{R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G - \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right)}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2} + i\frac{\frac{lc}{R(\hat{X})}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right) + \frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G^2}{\left(\frac{lc}{R(\hat{X})}\right)^2 + \left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}
$$

A4.4 Stability

The system is stable and the dynamics is dampening if:

$$
\frac{lc}{R(\hat{X})}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R(\hat{X})}\right) + \frac{4m^2 ca_0}{\left(\nabla_{\hat{X}}R(\hat{X})\right)^2}G^2 > 0 \tag{351}
$$

To study the sign of (351) we need to estimate the coefficient k.

A4.4.1 Estimation of the coefficients k, l and m

We can estimate k and l by computing the factors $C_i\left(p, \hat{X}\right)$, for $i = 1, 2, 3$.

This is done by estimating $p + \frac{1}{2}$. We start with the asymptotic form of $\hat{\Gamma}\left(p + \frac{1}{2}\right)$:

$$
\hat{\Gamma}\left(p + \frac{1}{2}\right) \simeq \sqrt{p + \frac{1}{2}}\exp\left(-\frac{\sigma_X^2\left(p + \frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)
$$

and rewriting the equation for $K_{\hat{X}}$ as:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|\left(\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)^{\frac{1}{4}}=C\left(\bar{p}\right)\sigma_{\hat{K}}^2\exp\left(-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\sqrt{\left(p+\frac{1}{2}\right)\sqrt{\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}}}$$

$$(352)$$

Then, using (343), we set:

$$\left(p+\frac{1}{2}\right)\sqrt{\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}}=\sqrt{C_1\left(p,\hat{X}\right)}$$

$$(353)$$

Equation (352) writes:

$$\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|\left(\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f(\hat{X})\right|^3}\right)^{\frac{1}{4}}}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}=\exp\left(-C_1\left(p,\hat{X}\right)\right)\left(C_1\left(p,\hat{X}\right)\right)^{\frac{1}{4}}$$

$$(354)$$

and the solution to (354) is:

$$C_1\left(p,\hat{X}\right)\quad=\quad\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}$$

$$(355)$$

$$=\quad C_0\left(\hat{X},K_{\hat{X}}\right)\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)$$

with:

$$C_0\left(\hat{X},K_{\hat{X}}\right)=\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}$$

and where $W\left(k,x\right)$ is the Lambert W function. The parameter $k=0$ for the stable case with low $K_{\hat{X}}$ and $k=-1$ for the unstable case with $K_{\hat{X}}$ large.

We can deduce $p+\frac{1}{2}$ from (355):

$$p+\frac{1}{2}=\frac{\sqrt{C_1\left(p,\hat{X}\right)}}{\sqrt{\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f(\hat{X})\right|^3}}}$$

$$(356)$$

and $2\frac{C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}$:

$$2\frac{C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}=\sqrt{C_1\left(p,\hat{X}\right)}\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}$$

From (356) and (??) we deduce:

$$C_2\left(p,\hat{X}\right)\quad=\quad\ln\left(p+\frac{1}{2}\right)-\frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}$$

$$(357)$$

$$=\quad\frac{1}{2}\ln\frac{C_1\left(p,\hat{X}\right)}{\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f(\hat{X})\right|^3}}-\sqrt{C_1\left(p,\hat{X}\right)}\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\simeq\frac{1}{2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

We can also compute:

$$\left(p+\frac{3}{2}\right)\ln\left(p+\frac{1}{2}\right) \simeq \frac{48\left|f\left(\hat{X}\right)\right|^3 C_1\left(p,\hat{X}\right)}{\sigma_X^2\left(f'\left(X\right)\right)^2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

$$= \frac{1}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

$$\times\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)$$

so that:

$$C_3\left(p,\hat{X}\right) = 1-C_1\left(p,\hat{X}\right)+\left(p+\frac{3}{2}\right)C_2\left(p,\hat{X}\right) \tag{358}$$

$$= 1-C_1\left(p,\hat{X}\right)+\left(p+\frac{3}{2}\right)\left(\ln\left(p+\frac{1}{2}\right)-\frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}\right)$$

$$\simeq 1+\left(\frac{48\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}-1\right)C_1\left(p,\hat{X}\right)$$

$$\simeq 1+\frac{1}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

Given that our assumptions $\sigma_X^2 < 1$ and in most cases $\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2(f'(X))^2} \gg 1$, then $\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2(f'(X))^2} \gg 1$ and $C_3\left(p,\hat{X}\right) \gg 1$.

These computations allow to estimate k and l. We start with k. Given that (see (346)):

$$k = 1-\eta\left(1-\frac{\gamma C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}$$

$$+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right)-(1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$

$$l = \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}$$

$$m = \left(1-\frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}$$

the sign of k and l depend on the magnitude of $K_{\hat{X}}$.

A4.4.1.1 $K_{\hat{X}} \gg 1$ For $K_{\hat{X}} \gg 1$, using (134) and:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 = D-\left(\nabla_X R\left(\hat{X}\right)\right)^2 K_{\hat{X}}^\alpha$$

we have:

$$K_{\hat{X}}^{\alpha} \simeq \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2} - \frac{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c}$$

and:

$$\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \simeq \frac{D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}}$$

The constant c has been defined in appendix 3, and satisfies $c << 1$. As a consequence:

$$\begin{aligned}
k &\simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} - (1-\alpha) \frac{C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\
&\simeq \left(\frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} - (1-\alpha)\right) \frac{C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}
\end{aligned}$$

This may be negative or positive depending on the relative magnitude of $\frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C(\bar{p})\sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R(\hat{X})\right)^{\frac{2}{\alpha}}}$ and $(1-\alpha)$. The first case correspond to the stable equilibrium with large $K_{\hat{X}}$ and the second case to the stable case with large $K_{\hat{X}}$ studied in appendix 2.

Unstable case This case corresponds to:

$$\frac{D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1$$

Moreover, using (358) and the following estimation, we have:

$$k \simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1 \tag{359}$$

We can also estimate $\left|\frac{k}{K_{\hat{X}}}\right|$. In this case:

$$\frac{k}{K_{\hat{X}}} \simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\eta\gamma D^{\frac{1}{\alpha}} c}{C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \sqrt{\frac{M-c}{c}} \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1 \tag{360}$$

We can estimate l by the same token:

$$l = \frac{\varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) C_3\left(p, \hat{X}\right)}{f\left(\hat{X}\right)} >> 1$$

and using (360) we have:

$$\left|\frac{k}{K_{\hat{X}}}\right| >> l$$

The coefficient m is obtained by using that in this case:

$$m \simeq \left(1 - \frac{\gamma C_3\left(p, \hat{X}\right)}{f\left(\hat{X}\right)}\right) \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \simeq -\frac{1}{\eta} k$$

Stable case For the stable case we have:

$$\frac{\eta\gamma D^{1+\frac{1}{\alpha}}c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}}-(1-\alpha)<0$$

and we write:

$$k\simeq-(1-\alpha)\frac{C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}<0$$

We have:

$$|k|>>1$$

and moreover:

$$\left|\frac{k}{K_{\hat{X}}}\right|\simeq(1-\alpha)\frac{C_3\left(p,\hat{X}\right)}{K_{\hat{X}}\left|f\left(\hat{X}\right)\right|}=\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)K_{\hat{X}}}l<<l \tag{361}$$

The coefficient m is obtained by using that in the stable case:

$$m\simeq-\frac{\gamma}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

A4.4.1.2 $K_{\hat{X}}<<1$ On the other hand, for $K_{\hat{X}}\leqslant 1$, we have:

$$\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}<<1 \tag{362}$$

so that:

$$k\simeq 1+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right)-(1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$

Given (357) and (358), this yields:

$$k\simeq-\frac{(1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}<0 \tag{363}$$

and, as in the previous case:

$$\begin{aligned}|k|&>>1\\l&>>1\end{aligned}$$

Moreover, given that $K_{\hat{X}}<<1$:

$$\left|\frac{k}{K_{\hat{X}}}\right|>>1 \tag{364}$$

and:

$$\left|\frac{k}{K_{\hat{X}}}\right|>>l \tag{365}$$

Moreover, given (362):

$$|m|=\left|1-\frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right|\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}<<\left|\frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right|$$

and:

$$|m|<<l$$

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A4.4.1.3 Intermediate case   In this case, we can consider that $\frac{D-\|\Psi(\hat{X})\|^2}{\|\Psi(\hat{X})\|^2}$ is of order 1:

$$\frac{D - \left\| \Psi\left(\hat{X}\right)\right\|^2}{\left\| \Psi\left(\hat{X}\right)\right\|^2} = O\left(1\right) \tag{366}$$

Assuming that $\gamma << 1$ we have:

$$k \simeq 1 + \frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right) - (1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$

$$\simeq 1 + \frac{\frac{\alpha}{2}\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right) - \frac{1-\alpha}{2}\left(\frac{K_{\hat{X}}\|\Psi(\hat{X})\|^2|f(\hat{X})|}{C(\bar{p})\sigma_{\hat{K}}^2}\right)^4 \exp\left(-W\left(k, -4C_0\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{\left|f\left(\hat{X}\right)\right|}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

Given that the intermediate case is stable (see appendix 2), the relation between $K_{\hat{X}}$ and $R\left(\hat{X}\right)$ is positive, we can assume that $k < 0$ and:

$$k \simeq 1 + \frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right) - (1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$

$$\simeq -\frac{\frac{1-\alpha}{2}\left(\frac{K_{\hat{X}}\|\Psi(\hat{X})\|^2|f(\hat{X})|}{C(\bar{p})\sigma_{\hat{K}}^2}\right)^4 \exp\left(-W\left(0, -4C_0\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{\left|f\left(\hat{X}\right)\right|}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

and:

$$l = \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)} \simeq l$$

$$= \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left(\frac{K_{\hat{X}}\|\Psi(\hat{X})\|^2|f(\hat{X})|}{C(\bar{p})\sigma_{\hat{K}}^2}\right)^4 \exp\left(-W\left(0, -4C_0\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{f\left(\hat{X}\right)}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

Note that in this case:

$$k \simeq -\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

and, given (366):

$$m \simeq -\gamma\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2 \varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

A4.4.2 Stability conditions

A4.4.2.1 Case $K_{\hat{X}} >> 1$

Stable case   As shown above, $k < 0$, $|k| >> 1$, $l >> 1$ and $\left| \frac{k}{K_{\hat{X}}} \right| << l$. Coefficients l and m are of the same order. Thus (351) becomes:

$$\frac{l^2 a_0 c}{\left(R\left(\hat{X}\right) \right)^2} + \frac{4m^2 c a_0}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right) \right)^2} G^2 > 0$$

That is, for $c > 0$ the oscillations are stable, whereas for $c < 0$ they are unstable.

Unstable case   In this case, $k > 0$, $|k| >> 1$, $l >> 1$ and $\left| \frac{k}{K_{\hat{X}}} \right| >> l$. We have also $m \simeq -\frac{1}{\eta}k$ and (351) writes:

$$\frac{cl}{R\left(\hat{X}\right)} \frac{k}{K_{\hat{X}}} + \frac{4k^2 c a_0}{\eta^2 \left(\nabla_{\hat{X}} R\left(\hat{X}\right) \right)^2} G^2 > 0 \tag{367}$$

That is, for $c > 0$ the oscillations are stable, whereas for $c < 0$ they are unstable.

A4.4.2.1.2 Case $K_{\hat{X}} << 1$ Equations (363) and (364) show that $k < 0$, $|k| >> 1$, $l >> 1$, $|m| << l$ and $\left| \frac{k}{K_{\hat{X}}} \right| >> l$. Equation (351) thus writes:

$$\frac{cl}{R\left(\hat{X}\right)} \frac{k}{K_{\hat{X}}} > 0 \tag{368}$$

That is, for $c > 0$ the oscillations are unstable, whereas for $c < 0$ they are stable.

A4.4.2.3 Intermediate case   In this case, we have seen above that $k < 0$:

$$k \simeq -\frac{1 - \alpha}{\varsigma F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right)} l$$

and:

$$m \simeq -\gamma \frac{D - \left\| \Psi\left(\hat{X}\right) \right\|^2}{\left\| \Psi\left(\hat{X}\right) \right\|^2 \varsigma F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right)}$$

As a consequence, equation (351) particularizes as:

$$\frac{l^2 c}{R\left(\hat{X}\right)} \left(\frac{a_0}{R\left(\hat{X}\right)} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right)} \right) + 4c a_0 \left(\frac{\gamma \left(D - \left\| \Psi\left(\hat{X}\right) \right\|^2 \right)}{\varsigma \nabla_{\hat{X}} R\left(\hat{X}\right) F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right) \left\| \Psi\left(\hat{X}\right) \right\|^2} \right)^2 G^2 > 0$$

Given the definition of a_0 and the stability of the intermediate case, we assume $a_0 > 0$. Thus, 2 possibilities arise.

Coefficient $c > 0$ In this case, the oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right)} > 0$$

or if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1 - \alpha}{\varsigma K_{\hat{X}} F_1 \left(R\left(K_{\hat{X}}, \hat{X} \right) \right)} < 0$$

and:

$$G^2 > \frac{l^2\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}{4a_0 R\left(\hat{X}\right)} \left(\frac{\varsigma\left(\nabla_{\hat{X}} R\left(\hat{X}\right) F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\gamma\left(D - \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}\right)^2 \left|\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\right|$$

the oscillations are unstable.

Coefficient $c < 0$ The oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma K_{\hat{X}} F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)} < 0$$

and:

$$G^2 < \frac{l^2\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}{4a_0 R\left(\hat{X}\right)} \left(\frac{\varsigma\left(\nabla_{\hat{X}} R\left(\hat{X}\right) F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\gamma\left(D - \left\|\Psi\left(\hat{X}\right)\right\|^2\right)}\right)^2 \left|\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\right|$$

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